Segal-Bargmann transforms associated with Coxeter groups

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SEGAL-BARGMANN TRANSFORMS ASSOCIATED WITH COXETER GROUPS

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ABSTRACT. In this paper we present a generalization of the Segal-Bargmann transform associated with finite reflection groups on $\mathbb{R}^N$. We give the integral kernel appearing in the generalized Segal-Bargmann transform and we prove the unitarity of this transform. To define the above mentioned transform, we introduce a generalized Fock space $\mathscr{F}_k(\mathbb{C}^N)$ on $\mathbb{C}^N$ with reproducing kernel the Dunkl-kernel. The definition and properties of $\mathscr{F}_k(\mathbb{C}^N)$ extend naturally those of its classical counterpart $\mathscr{F}_0(\mathbb{C}^N)$. The Segal-Bargmann transform gives the analogue of the Dunkl theory in the Fock model.

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Contents

1. INTRODUCTION

The study of several generalizations of the classical Segal-Bargmann transform has a long and rich history in many different settings [1, 32, 23, 17, 7, 8, 34]. It is well-known that the classical Segal-Bargmann transform maps unitarily from the Schödinger model to the Fock model intertwining the action of the Heisenberg groups. There are many ways of computing the integral kernel appearing in the Segal-Bargmann transform and showing the unitarity of this transform. One unifying tool is the restriction principle, i.e. polarization of a suitable restriction map [22, 23]. This idea uses the heat-kernel analysis.

While the theory of Segal-Bargmann transform has been pursued for a long time, the growing interest in Dunkl theory and special functions related to Coxeter groups is comparably recent. However, there has been a rapid development in this area in the last few years. Among the broad literature in this area, we refer to [10, 26, 18, 21, 29, 3], and references therein.

In the present paper, we employ the restriction principle to construct the Segal-Bargmann transform associated with finite Coxeter groups. This suggests to introduce and study new Fock-type spaces which generalize the classical Bargmann-Fock model [1]. To realize the integral representation of the Segal-Bargmann transform, we use Rösler's results on the heat-kernel associated with reflection groups [30].

The motivation for studying the Segal-Bargmann transform is to exhibit some relationships between Dunkl's theory and its applications in the Schrödinger model and in the Fock model, for instance, the study of the Dunkl operators, the Calogero-Moser systems, and the Dunkl transform. It turns out that the Dunkl transform in the Fock model, is the dilatation operator on functions by the complex number $-i$. This assertion gives an alternative and simple proof of the unitarity of the Dunkl transform, which was investigated earlier independently in [12, 21].

To be more specific about our results, let G be a finite Coxeter group on $\mathbb{R}^N$ with root system R, and let $k : R \to \mathbb{R}^+$ be a non-negative multiplicity function. The Dunkl operators are defined by

$$T_\xi(k)f(x) = \partial_\xi f(x) + \sum_{\alpha\in R^+} k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, x\rangle}(f(x) - f(r_\alpha x)), \qquad x, \xi \in \mathbb{R}^N,$$

where R^+ is a positive subsystem of R, $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product in $\mathbb{R}^N$, and r_α is the reflection on the hyperplane orthogonal to α. If the multiplicity function $k \equiv 0$, then $T_\xi(k)$ coincides with the partial derivative ∂_ξ. An important ingredient in the theory of Dunkl operators is the generalized exponential kernel $E_k(\cdot, \cdot)$ on $\mathbb{R}^N \times \mathbb{R}^N$, which can be characterized as the unique solution of a joint eigenfunction problem for the Dunkl operators $\{T_\xi(k) \mid \xi \in \mathbb{R}^N\}$ with the initial condition $E_k(\cdot, 0) = 1$ (cf. [11, 26]). In particular, $E_0(x, y) = e^{\langle x,y\rangle}$.

In the first part of the present paper, we prove that there exists a Hilbert space $\mathscr{F}_k(\mathbb{C}^N)$ of holomorphic functions on $\mathbb{C}^N$ with reproducing kernel $E_k(z, \bar{w})$, for $z, w \in \mathbb{C}^N$, holomorphic in z and anti-holomorphic in w. Further, we show that $\mathscr{P}(\mathbb{C}^N)$, the algebra of polynomial functions on $\mathbb{C}^N$, is contained in $\mathscr{F}_k(\mathbb{C}^N)$ as a dense subspace. Further, if we denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_k$ the inner product in $\mathscr{F}_k(\mathbb{C}^N)$, we obtain the following Fischer-type formula

$$\langle\!\langle p, q \rangle\!\rangle_k = p(T)\overline{q(\bar{z})}\Big|_{z=0}, \qquad p, q \in \mathscr{P}(\mathbb{C}^N).$$

We conclude the first part of the paper by proving that the operators $T_\xi(k)$ and M_ξ, where M_ξ is the multiplication operator $M_\xi f(z) = \langle z, \xi\rangle f(z)$ for $z, \xi \in \mathbb{C}^N$ and $f \in \mathscr{F}_k(\mathbb{C}^N)$, are closed densely defined on $\mathscr{F}_k(\mathbb{C}^N)$, such that

$$\langle\!\langle T_\xi(k)f, g \rangle\!\rangle_k = \langle\!\langle f, M_\xi g \rangle\!\rangle_k, \qquad f, g \in \mathscr{F}_k(\mathbb{C}^N),$$

whenever the both sides of the equation make sense.

The classical Bargmann-Fock space corresponds to $\mathscr{F}_0(\mathbb{C}^N)$ (cf. [1]).

The second part of the present paper deals with the generalized Segal-Bargmann transform associated with G, and its applications.

Let $\mathscr{L}^2(\mathbb{R}^N, w_k)$ be the space of $\mathscr{L}^2$-functions on $\mathbb{R}^N$ with respect to the weighted measure $w_k(x)dx = \prod_{\alpha\in R^+} |\langle\alpha, x\rangle|^{2k_\alpha} dx$. This $\mathscr{L}^2$-space plays an important role in the Dunkl theory. By taking a Gaussian multiplier into account, we get a bounded injective map $\mathscr{R}_k : \mathscr{F}_k(\mathbb{C}^N) \to \mathscr{L}^2(\mathbb{R}^N, w_k)$ with dense image. Let $\mathscr{R}_k^* = \sqrt{\mathscr{R}_k\mathscr{R}_k^*}\mathscr{B}_k$ be the polar decomposition of $\mathscr{R}_k^*$. The map $\mathscr{B}_k$ is the so-called generalized Segal-Bargmann transform. Using the heat-kernel analysis associated with reflection groups [30], we prove that the integral representation of $\mathscr{B}_k$ is given by

$$\mathscr{B}_k f(z) = c(k)e^{-\langle z,z\rangle/2} \int_{\mathbb{R}^N} f(x)E_k(\sqrt{2}x, \sqrt{2}z)e^{-\langle x,x\rangle} w_k(x)dx, \qquad z \in \mathbb{C}^N,$$

for some explicit constant $c(k)$. The transform $\mathscr{B}_k$ is a unitary isomorphism from $\mathscr{L}^2(\mathbb{R}^N, w_k)$ to $\mathscr{F}_k(\mathbb{C}^N)$. Moreover, the following diagram commutes

$$\begin{array}{ccc} \mathscr{L}^2(\mathbb{R}^N, w_k) & \xrightarrow{\mathscr{B}_k} & \mathscr{F}_k(\mathbb{C}^N) \\ {\scriptstyle \mathscr{D}_k}\downarrow & & \downarrow{\scriptstyle (-i)^*} \\ \mathscr{L}^2(\mathbb{R}^N, w_k) & \xrightarrow{\mathscr{B}_k} & \mathscr{F}_k(\mathbb{C}^N) \end{array}$$

where $\mathscr{D}_k$ is the Dunkl transform (or the generalized Fourier transform), and $(-i)^*F(z) = F(-iz)$ for $F \in \mathscr{F}_k(\mathbb{C}^N)$. The above statement gives an alternative and simple proof for the unitarity of $\mathscr{D}_k$, studied earlier by Dunkl [12] and de Jeu [21].

Finally, we exhibit the relationship between operators on $\mathscr{L}^2(\mathbb{R}^N, w_k)$ and on $\mathscr{F}_k(\mathbb{C}^N)$ by means of the Segal-Bargmann transform $\mathscr{B}_k$. For instance, on $\mathscr{F}_k(\mathbb{C}^N)$, the gauge equivalent version of the Hamiltonian of the Calogero-Moser system with harmonic confinement has the form

$$\check{\mathscr{H}}_k = \gamma + N/2 + \sum_{i=1}^{N} \xi_i \partial_{\xi_i}.$$

Here $\gamma := \sum_{\alpha \in R^+} k_\alpha$, and $\{\xi_1, \ldots, \xi_N\}$ is any orthonormal basis of $\mathbb{C}^N$. Now, the spectral properties of $\check{\mathscr{H}}_k$ are rather easy to describe and it is possible to obtain complete bases of eigenfunctions.

Our setting includes the case of Segal-Bargmann transform associated with flat symmetric spaces, where the Coxeter group G becomes the Weyl group related to the symmetric space. In [34], the author considers such a transformation, associated with the flat symmetric spaces of type C_N and D_N ($N \geq 3$), for an invariant subspace of $\mathscr{F}_0(\mathbb{C}^N)$.

For the rank one case, i.e. $N = 1$, Cholewinski [5] has investigated the Segal-Bargmann transform only on the Hilbert space of even entire functions in $\mathscr{F}_k(\mathbb{C})$, by employing another approach.

This paper is organized as follows. In Section 2 we give an abbreviated background on the Dunkl theory. Section 3 is devoted to the study of the Fock space $\mathscr{F}_k(\mathbb{C}^N)$. In Section 4 we turn our attention to the Segal-Bargmann transform and its applications. Section 5 deals with the Weyl quantization map and the Berezin transform associated with the Coxeter group G. We establish the integral representation of the Berezin transform and we prove, abstractly, that the Weyl quantization map is a unitary isomorphism from $\mathscr{L}^2(\mathbb{R}^N, w_k) \otimes \mathscr{L}^2(\mathbb{R}^N, w_k)$ to $\mathscr{F}_k(\mathbb{C}^N) \otimes \overline{\mathscr{F}_k(\mathbb{C}^N)}$. It should be interesting to pursue this quantization further, as well as the possibility of studying the "field theory" case of $N \to \infty$.

2. Background

For $\alpha \in \mathbb{R}^N \setminus \{0\}$, denote by r_α the reflection on the hyperplane $\langle \alpha \rangle^\perp$ orthogonal to α

$$r_\alpha(x) = x - 2\frac{\langle \alpha, x \rangle}{|\alpha|^2}\alpha, \qquad x \in \mathbb{R}^N,$$

where $\langle x, y \rangle = \sum_{i=1}^N x_i y_i$ and $|x| = \sqrt{\langle x, x \rangle}$. The reflection r_α belongs to the orthogonal group $O(N)$. We will use the same notation $\langle \cdot, \cdot \rangle$ for the bilinear extension of the Euclidean scalar product to $\mathbb{C}^N \times \mathbb{C}^N$.

A finite set $R \subset \mathbb{R}^N \setminus \{0\}$ is called a root system if

$$R \cap \mathbb{R}\alpha = \{\pm\alpha\}, \qquad \forall \alpha \in R,$$
$$r_\alpha(R) = R, \qquad \forall \alpha \in R.$$

Henceforth, we will assume that the root system R is normalized in the sense that $|\alpha|^2 = 2$ for all $\alpha \in R$. This simplifies formulas, with no loss of generality for our purposes.

A Coxeter group G is a finite subgroup of $O(N)$ generated by the reflections $\{ r_\alpha \mid \alpha \in R \}$. A multiplicity function on R is a G-invariant function $k : R \to \mathbb{C}$. Setting $k_\alpha := k(\alpha)$ for $\alpha \in R$, we have $k_{g\alpha} = k_\alpha$ for all $g \in G$. The $\mathbb{C}$-vector space of multiplicity functions on R is denoted by $\mathscr{K}$. If $m = \sharp\{G\text{-orbits in } R\}$, then $\mathscr{K} \cong \mathbb{C}^m$.

Let R^+ be a choice of positive roots in R. For $\xi \in \mathbb{R}^N$ and $k = (k_\alpha)_{\alpha \in R} \in \mathscr{K}$, the Dunkl operator $T_\xi(k)$ is defined by

$$T_\xi(k)f(x) = \partial_\xi f(x) + \sum_{\alpha \in R^+} k_\alpha \langle \alpha, \xi \rangle \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle}, \qquad f \in \mathscr{C}^1(\mathbb{R}^N).$$

Here ∂_ξ denotes the directional derivative corresponding to ξ. The definition of $T_\xi(k)$ is independent of the choice of R^+, and it is a homogeneous differential operator of degree -1. Moreover, by the G-invariance of the multiplicity function k, $T_\xi(k)$ satisfies

$$g_0 \circ T_\xi(k) \circ g_0^{-1} = T_{g_0 \xi}(k), \qquad \forall\, g_0 \in G.$$

Further, if f and g in $\mathscr{C}^1(\mathbb{R}^N)$, and at least one of them is G-invariant, then

$$T_\xi(k)(fg) = T_\xi(k)(f)g + fT_\xi(k)(g). \tag{2.1}$$

The remarkable property of the Dunkl operators is that the family $\{T_\xi(k), \xi \in \mathbb{R}^N\}$ generates a commutative algebra of linear operators on the $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^N$. For more details on the Dunkl operators we refer to [10, 11, 12], and references therein.

For any orthonormal basis $\{\xi_1, \ldots, \xi_N\}$ of $\mathbb{R}^N$, set

$$\Delta_k = \sum_{i=1}^N T_{\xi_i}(k)^2.$$

The generalized Laplacian Δ_k is homogeneous of degree -2. By the normalization $\langle \alpha, \alpha \rangle = 2$, we can rewrite Δ_k as

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R^+} k_\alpha \left\{ \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle^2} \right\},$$

where Δ and ∇ denote the usual Laplacian and gradient, respectively. For all i-th basis vector ξ_i, we will use the abbreviation $T_{\xi_i}(k) = T_i(k)$.

Denote by $\mathscr{P}(\mathbb{R}^N) = \mathbb{C}[\mathbb{R}^N]$ the $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^N$, and by $\mathscr{P}_n$, for $n \in \mathbb{Z}_+$, the subspace of homogeneous polynomials of degree n. Next we will use the following notations: for $\mathbf{m} = (m_1, \ldots, m_N) \in \mathbb{Z}_+^N$, write

$$\mathbf{m}! = m_1! \cdots m_N!, \quad |\mathbf{m}| = m_1 + \cdots + m_N, \quad z^{\mathbf{m}} = z_1^{m_1} \cdots z_N^{m_N}, \quad T^{\mathbf{m}} = T_1(k)^{m_1} \cdots T_N(k)^{m_N},$$

where $z \in \mathbb{C}^N$ and $T = (T_1(k), \ldots, T_N(k))$ is our family of commuting Dunkl operators on $\mathscr{P}(\mathbb{R}^N)$.

In [11], Dunkl proved that for $k \geq 0$, there exists an intertwining operator relating $T_\xi(k)$ to the usual partial differential operators. This result was generalized later by Dunkl, de Jeu, and Opdam in [14] to a more general setting.

Theorem 2.1. (cf. [14]) *Let* $\mathscr{K}^{\mathrm{reg}} = \{ k \in \mathscr{K} \mid \cap_{\xi \in \mathbb{R}^N} \operatorname{Ker}(T_\xi(k)) = \mathbb{C} \cdot 1\}$. *The following assertions are equivalent:*

(i) $k \in \mathscr{K}^{\mathrm{reg}}$,

(ii) *there exists a unique intertwining operator* V_k *on* $\mathscr{P}(\mathbb{R}^N)$ *such that*

$$V_k(\mathscr{P}_n) \subset \mathscr{P}_n, \quad V_k\big|_{\mathscr{P}_0} = \mathrm{id}, \quad T_\xi(k) V_k = V_k \partial_\xi \quad (\xi \in \mathbb{R}^N).$$

An explicit formula for V_k is still an open problem; it is only for $G = (\mathbb{Z}/2\mathbb{Z})^N$ and S_3 where the explicit form of V_k is known (cf. [13, 33]).

Notice that $\{k \in \mathscr{K} \mid k \geq 0\} \subset \mathscr{K}^{\mathrm{reg}}$. In most parts of the paper we will restrict our attention to non-negative multiplicity functions.

From Theorem 2.1, it follows that for $k \in \mathscr{K}^{\mathrm{reg}}$, there exists a unique bijective linear map $\tilde{V}_k : \mathscr{P}(\mathbb{R}^N) \to \mathscr{P}(\mathbb{R}^N)$ such that

$$\tilde{V}_k V_k = V_k \tilde{V}_k = \mathrm{id}, \quad \tilde{V}_k(\mathscr{P}_n) \subset \mathscr{P}_n, \quad \tilde{V}_k\big|_{\mathscr{P}_0} = \mathrm{id}, \quad \partial_\xi \tilde{V}_k = \tilde{V}_k T_\xi(k) \quad (\xi \in \mathbb{R}^N).$$

In particular, for $p \in \mathscr{P}(\mathbb{R}^N)$ and $k \in \mathscr{K}^{\mathrm{reg}}$, one can show that

$$\tilde{V}_k p(x) = \sum_{n=0}^{\infty} \sum_{|\mathbf{m}|=n} \frac{x^{\mathbf{m}}}{\mathbf{m}!} T(k)^{\mathbf{m}} p(0).$$

Therefore, for all analytic functions f in a neighborhood of 0 and for $k \in \mathscr{K}^{\mathrm{reg}}$

$$f(z) = \sum_{n=0}^{\infty} \sum_{|\mathbf{m}|=n} \frac{V_k(z^{\mathbf{m}})}{\mathbf{m}!} T(k)^{\mathbf{m}} f(0). \tag{2.2}$$

For $k \in \mathscr{K}^{\mathrm{reg}}$, there exists a generalization of the usual exponential kernel $e^{\langle \cdot, \cdot \rangle}$ by means of the Dunkl system of differential equations.

Theorem 2.2. (cf. [26]) *There exists a unique meromorphic function* E_k *on* $\mathbb{C}^N \times \mathscr{K} \times \mathbb{C}^N$ *characterized by:*

(i) $T_\xi(k) E_k(z, w) = \langle \xi, w \rangle E_k(z, w)$; *and*

(ii) $E_k(z, 0) = 1$.

Moreover, this function satisfies

(iii) E_k *is holomorphic on* $\mathbb{C}^N \times (\mathscr{K} \setminus \mathscr{K}^{\mathrm{reg}}) \times \mathbb{C}^N$; *and*

(iv) $E_k(g_0 \cdot z, g_0 \cdot w) = E_k(z, w)$ *for all* $g_0 \in G$.

The function E_k is the so-called Dunkl-kernel, or the k-exponential function. For $k \equiv 0$, we have $E_0(z, w) = e^{\langle z, w \rangle}$ for $z, w \in \mathbb{C}^N$.

As E_k is a holomorphic function on $\mathbb{C}^N \times \mathbb{C}^N$, by (2.2) one can obtain its Taylor series as

$$E_k(z,w) = \sum_{n=0}^{\infty} \sum_{|\mathbf{m}|=n} \frac{V_k(z^{\mathbf{m}})}{\mathbf{m}!} \big[T^{\mathbf{m}}(k) E_k(z,w)\big]\Big|_{z=0}, \qquad z, w \in \mathbb{C}^N$$

with $T_\xi^{\mathbf{m}}(k)E_k(z,w) = \langle \xi, w\rangle^{\mathbf{m}} E_k(z,w)$. Therefore

$$E_k(z,w) = \sum_{n=0}^{\infty} E_k^{(n)}(z,w), \quad \text{with} \quad E_k^{(n)}(z,w) = \sum_{|\mathbf{m}|=n} \frac{V_k(z^{\mathbf{m}})}{\mathbf{m}!} w^{\mathbf{m}}. \tag{2.3}$$

We close this section by two Macdonald-type identities for the Dunkl-kernel E_k.

For $k \geq 0$, let w_k be the weight function on $\mathbb{R}^N$ defined by

$$w_k(x) = \prod_{\alpha \in R^+} |\langle \alpha, x\rangle|^{2k_\alpha}.$$

For all $g_0 \in G$ and all $\lambda \in \mathbb{C}$, we have $w_k(g_0 x) = w_k(x)$ and $w_k(\lambda x) = \lambda^{2\gamma} w_k(x)$, with $\gamma := \sum_{\alpha \in R^+} k_\alpha$. Further, let

$$c_k := \int_{\mathbb{R}^N} e^{-|x|^2/2} w_k(x) dx,$$

which is called the Macdonald-Metha-Selberg integral. In [26] Opdam gives a closed form for c_k for finite Coxeter groups.

The following proposition is crucial in Dunkl's theory and its applications.

Proposition 2.3. (cf. [12]) *For non-negative multiplicity function* k*, and for* $p \in \mathscr{P}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} e^{-\Delta_k/2} p(x) E_k(x,w) e^{-|x|^2/2} w_k(x) dx = c_k e^{\langle w,w\rangle/2} p(w), \quad w \in \mathbb{C}^N, \tag{2.4}$$

$$\int_{\mathbb{R}^N} E_k(x,z) E_k(x,w) e^{-|x|^2/2} w_k(x) dx = c_k e^{(\langle z,z\rangle + \langle w,w\rangle)/2} E_k(z,w), \quad z, w \in \mathbb{C}^N. \tag{2.5}$$

3. FOCK SPACES ASSOCIATED WITH COXETER GROUPS

For the reader's convenience, let us recall the definition of a reproducing kernel. Let $\mathscr{H}$ be a Hilbert space whose elements are complex-valued functions on a set S. A reproducing kernel for $\mathscr{H}$ is a complex-valued function $\mathbb{K}$ on $S \times S$ such that, denoting $\mathbb{K}_w(z) = \mathbb{K}(z,w)$, $\mathbb{K}_w$ belongs to $\mathscr{H}$ for all w, and $f(w) = \langle\langle f, \mathbb{K}_w \rangle\rangle$ for all functions f in $\mathscr{H}$ and all w in S.

For $z, w \in \mathbb{C}^N$, define

$$\mathbb{K}_{k,w}(z) = \mathbb{K}_k(z,w) := E_k(z, \bar{w}).$$

As k will be fixed, we will write $\mathbb{K}$ for $\mathbb{K}_k$.

Theorem 3.1. (i) *The kernel* $\mathbb{K}(z,w)$ *is a positive definite kernel, i.e. for all* $z^{(1)}, \ldots, z^{(\ell)} \in \mathbb{C}^N$ *and* $\alpha_1, \ldots \alpha_\ell \in \mathbb{C}$

$$\sum_{i,j=1}^{\ell} \alpha_i \overline{\alpha_j} \mathbb{K}(z^{(i)}, z^{(j)}) \geq 0. \tag{3.1}$$

(ii) *The kernel* $\mathbb{K}$ *is continuous, and* $\mathbb{K}_w$ *is holomorphic for all* $w \in \mathbb{C}^N$.

(iii) *For all* $z, w \in \mathbb{C}^N$, $\mathbb{K}(z,w) = \overline{\mathbb{K}(w,z)}$.

Proof. (i) By the integral formula (2.5), we have

$$\begin{aligned}
&\sum_{i,j=1}^{\ell} \alpha_i \overline{\alpha_j} \mathbb{K}(z^{(i)}, z^{(j)}) \\
&= c_k^{-1} \sum_{i,j=1}^{\ell} \alpha_i \overline{\alpha_j} e^{-\langle z^{(i)}, z^{(i)} \rangle/2} e^{-\langle \overline{z^{(j)}}, \overline{z^{(j)}} \rangle/2} \int_{\mathbb{R}^N} E_k(x, z^{(i)}) E_k(x, \overline{z^{(j)}}) e^{-|x|^2/2} w_k(x)\, dx \\
&= c_k^{-1} \int_{\mathbb{R}^N} \Big\{ \sum_{i=1}^{\ell} \alpha_i E_k(x, z^{(i)}) e^{-\langle z^{(i)}, z^{(i)} \rangle/2} \Big\} \Big\{ \sum_{j=1}^{\ell} \overline{\alpha_j} E_k(x, \overline{z^{(j)}}) e^{-\langle \overline{z^{(j)}}, \overline{z^{(j)}} \rangle/2} \Big\} e^{-|x|^2/2} w_k(x)\, dx \\
&= c_k^{-1} \int_{\mathbb{R}^N} \Big| \sum_{i=1}^{\ell} \alpha_i E_k(x, z^{(i)}) e^{-\langle z^{(i)}, z^{(i)} \rangle/2} \Big|^2 e^{-|x|^2/2} w_k(x)\, dx \geq 0.
\end{aligned}$$

(ii) The second statement follows from the fact that E_k has a holomorphic extension to $\mathbb{C}^N \times \mathbb{C}^N$.

(iii) For $w \in \mathbb{C}^N$, the function $\overline{\mathbb{K}_{\bar{w}}}$ satisfies $T_\xi^x(k)\overline{\mathbb{K}_{\bar{w}}(x)} = \langle \xi, \bar{w} \rangle \overline{\mathbb{K}_{\bar{w}}(x)}$, for $\xi \in \mathbb{R}^N$, and $\overline{\mathbb{K}_{\bar{w}}}(0) = 1$. Here the superscript x denotes that the operator acts with respect to the x-variable. By the uniqueness of the solution of the Dunkl system of differential equations, it follows that $\overline{\mathbb{K}_{\bar{w}}(x)} = E_k(x, \bar{w})$, i.e. $\overline{\mathbb{K}_{\bar{w}}(x)} = \mathbb{K}_w(x)$ for all $x \in \mathbb{R}^N$. Since $x \mapsto \overline{\mathbb{K}_{\bar{w}}(\bar{x})}$ and $x \mapsto \mathbb{K}_w(x)$ are holomorphic on $\mathbb{C}^N$ and agree on $\mathbb{R}^N$, $\mathbb{K}(z,w) = \overline{\mathbb{K}(\bar{z}, \bar{w})}$. On the other hand, since the Dunkl kernel E_k is symmetric [12], then $\mathbb{K}(z,w) = \overline{K(\bar{z}, \bar{w})} = \overline{E_k(\bar{z}, w)} = \overline{E_k(w, \bar{z})} = \overline{K(w,z)}$. □

One may interpret the condition (3.1) as following: The kernel $\mathbb{K} : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ is positive definite if and only if for all $\ell \in \mathbb{N}$ and $z^{(1)}, \dots, z^{(\ell)} \in \mathbb{C}^N$, the matrices $\big(\mathbb{K}(z^{(i)}, z^{(j)})\big)_{1 \leq i,j \leq \ell}$ are positive elements of $M(\ell, \mathbb{C})$.

The following theorem gives a concrete meaning of the abstract concept of a positive definite kernel.

Theorem 3.2. *There exists a Hilbert space* $\mathscr{F}_k(\mathbb{C}^N)$ *of holomorphic functions, such that* $\mathbb{K}$ *is its reproducing kernel.*

Proof. Let $\mathbb{S}_k(\mathbb{C}^N)$ be the set of all finite complex linear combinations

$$f = \sum_{i=1}^{n} \alpha_i \mathbb{K}_{z^{(i)}}, \qquad \alpha_i \in \mathbb{C}, \quad z^{(i)} \in \mathbb{C}^N.$$

On $\mathbb{S}_k(\mathbb{C}^N)$, a Hermitian bilinear form can be defined by

$$\langle\langle f, g \rangle\rangle_k = \sum_{i=1}^{n} \sum_{j=1}^{\ell} \alpha_i \overline{\beta_j} \mathbb{K}(w^{(j)}, z^{(i)}),$$

with $g = \sum_{j=1}^{\ell} \beta_j \mathbb{K}_{w^{(j)}}$. Therefore $\langle\langle f, f\rangle\rangle_k \geq 0$ for all $f \in \mathbb{S}_k(\mathbb{C}^N)$. Moreover, $\langle\langle f, \mathbb{K}_w\rangle\rangle_k = f(w)$, and therefore

$$|f(w)|^2 \leq \langle\langle f, f\rangle\rangle_k \langle\langle \mathbb{K}_w, \mathbb{K}_w\rangle\rangle_k = \|f\|_k^2 \mathbb{K}(w, w).$$

Hence, if $\|f\|_k = 0$ then $f \equiv 0$, and the form $\langle\langle \cdot, \cdot\rangle\rangle_k$ is positive definite. Let $\mathscr{F}_k(\mathbb{C}^N)$ be the completion of $\mathbb{S}_k(\mathbb{C}^N)$ with respect to the norm $\|f\|_k^2 = \langle\langle f, f\rangle\rangle_k$.
Let (f_m) be a Cauchy sequence for this norm. Then

$$|f_p(z) - f_q(z)|^2 \leq \|f_p - f_q\|^2 \mathbb{K}(z, z) \qquad \forall\, z \in \mathbb{C}^N.$$

Therefore, the sequence is pointwise convergent. Moreover, since $w \mapsto \mathbb{K}(\cdot, w)$ is continuous, the equality

$$\langle\langle f, \mathbb{K}_w\rangle\rangle_k = f(w)$$

holds for all $f \in \mathscr{F}_k(\mathbb{C}^N)$, and therefore $\mathbb{K}$ is the reproducing kernel of the Hilbert space of functions $\mathscr{F}_k(\mathbb{C}^N)$.
The fact that $\mathscr{F}_k(\mathbb{C}^N)$ is contained in the set of holomorphic functions follows from the following: By Theorem 3.1(ii), $\mathbb{K}_w$ is holomorphic for all $w \in \mathbb{C}^N$. Hence, in the construction above, each f is holomorphic. Moreover, by Hartogs's theorem, it follows that $\mathbb{K}$ is continuous on $\mathbb{C}^N \times \mathbb{C}^N$ since $\mathbb{K}_w$ is holomorphic and therefore $\mathbb{K}$ is a holomorphic function of $(z, \bar{w})$ on $\mathbb{C}^N \times \mathbb{C}^N$. Thus, if (f_m) converges to f in the norm of $\mathscr{F}_k(\mathbb{C}^N)$, then it converges uniformly on compact subsets of $\mathbb{C}^N$ by

$$|f_m(z) - f(z)| \leq \|f_m - f\| \sqrt{\mathbb{K}(z, z)}.$$

□

From the above proposition, $\mathscr{F}_k(\mathbb{C}^N)$ is defined by

$$\mathscr{F}_k(\mathbb{C}^N) = \overline{\langle \mathbb{K}_z \mid z \in \mathbb{C}^N\rangle}.$$

Here the bar means the completion with respect to the norm $\|\cdot\|_k$. The Hilbert space $\mathscr{F}_k(\mathbb{C}^N)$ is uniquely determined by its reproducing kernel $\mathbb{K}$. Notice that, for $k \equiv 0$, the space $\mathscr{F}_0(\mathbb{C}^N)$ reduces to the classical Fock space of holomorphic functions f on $\mathbb{C}^N$ such that

$$\|f\|_0^2 := \pi^{-N} \int_{\mathbb{C}^N} |f(z)|^2 e^{-\|z\|^2} dz < \infty,$$

where $\|z\|^2 = \sum_{i=1}^N |z_i|^2$ (cf. [1]).

Lemma 3.3. *For non-negative multiplicity function* k

(i) $\langle\langle f, g\rangle\rangle_k = \overline{\langle\langle g, f\rangle\rangle_k}$ *for all* $f, g \in \mathscr{F}_k(\mathbb{C}^N)$,

(ii) $\langle\langle g_0 \cdot f, g_0 \cdot g\rangle\rangle_k = \langle\langle f, g\rangle\rangle_k$ *for all* $g_0 \in G$ *and all* $f, g \in \mathscr{F}_k(\mathbb{C}^N)$.

Proof. (i) Let $f, g \in \mathbb{S}_k(\mathbb{C}^N)$ such that $\langle\langle f, g\rangle\rangle_k = \sum_{i=1}^{\ell}\sum_{j=1}^{n} \alpha_i\overline{\beta_j}\mathbb{K}(w^{(j)}, z^{(i)})$. Using the fact that $\mathbb{K}(z, w) = \overline{\mathbb{K}(w, z)}$, we get

$$\langle\langle f, g\rangle\rangle_k = \sum_{i=1}^{\ell}\sum_{j=1}^{n} \overline{\beta_j\overline{\alpha_i}\mathbb{K}(z^{(i)}, w^{(j)})} = \overline{\langle\langle g, f\rangle\rangle_k}.$$

Now the statement holds by density.
(ii) The assertion follows from the fact that $\mathbb{K}(g_0 z, g_0 w) = \mathbb{K}(z, w)$ by employing the same argument used in (i). □

The following (standard) lemma will be useful later in several places.

Lemma 3.4. *In a normed space $\mathbb{X}$, a sequence $(x_n)_n$ converges weakly to $x \in \mathbb{X}$ if and only if:*
(i) *the sequence $(\|x_n\|)_n$ is bounded, and*
(ii) *for every element f of a total subset M in the dual $\mathbb{X}^*$ of $\mathbb{X}$, we have $(f(x_n))_n$ converges to $f(x)$.*

Proof. We sketch the proof. Suppose (i) and (ii) hold. Consider any $f \in \mathbb{X}^*$ and show that $f(x_n)$ converges to $f(x)$, which means weak convergence. By (i) we have $\|x_n\| \leq c$ for all n and $\|x\| \leq c$, where c is sufficiently large. Since M is total in $\mathbb{X}^*$, for every $f \in \mathbb{X}^*$ there is a sequence $(f_j)_j$ in $\langle M\rangle$ such that $f_j \to f$. Hence for any given $\epsilon > 0$ we can find a j such that $\|f - f_j\| < \frac{\epsilon}{3c}$. Moreover, since $f_j \in \langle M\rangle$, by (ii) there is a n_0 such that, for all $n > n_0$, $|f_j(x_n) - f_j(x)| < \frac{\epsilon}{3}$. Using these two inequalities, we obtain for $n > n_0$

$$\begin{aligned} |f(x_n) - f(x)| &\leq |f(x_n) - f_j(x_n)| + |f_j(x_n) - f_j(x)| + |f_j(x) - f(x)| \\ &< \|f - f_j\|\|x_n\| + \frac{\epsilon}{3} + \|f_j - f\|\|x\| < \epsilon. \end{aligned}$$

Therefore, the sequence $(x_n)_n$ converges weakly to x. The converse direction of the lemma is rather clear and will be omitted. □

Corollary 3.5. *If the above hypotheses (i), (ii) hold, and if in addition $\lim_{n\to\infty}\|x_n\| = \|x\|$, then the sequence $(x_n)_n$ converges strongly to x in $\mathbb{X}$.*

It is worthwhile to mention that if $\mathscr{H}$ is a Hilbert space, every $f \in \mathscr{H}^*$ has a Riesz representation $f(x) = \langle\langle x, h\rangle\rangle$ with $h \in \mathscr{H}$.

Proposition 3.6. *For $k \geq 0$, the Hilbert space $\mathscr{F}_k(\mathbb{C}^N)$ contains the $\mathbb{C}$-algebra $\mathscr{P}(\mathbb{C}^N)$ of polynomial functions on $\mathbb{C}^N$ as a dense subspace.*

Proof. Recall that $E_k(z, \bar{w}) \in \mathscr{F}_k(\mathbb{C}^N)$, since it is its reproducing kernel. In particular $1 = E(z, 0) \in \mathscr{F}_k(\mathbb{C}^N)$, and therefore the constants belong to $\mathscr{F}_k(\mathbb{C}^N)$. Let us now prove that for all $\mathbf{m} \in \mathbb{Z}_+^N$ and $z \in \mathbb{C}^N$, $z^{\mathbf{m}} \in \mathscr{F}_k(\mathbb{C}^N)$.

Notice that for $x, \xi \in \mathbb{R}^N$ and $z \in \mathbb{C}^N$

$$\langle \xi, z\rangle E_k(z,x) = \partial_\xi E_k(z,x) + \sum_{\alpha\in R^+} k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, x\rangle}(E_k(z,x) - E_k(z, r_\alpha x)).$$

We may assume $|\xi| = 1$. Since the function

$$z \mapsto \psi(t,z,x) := \frac{E_k(z, x+t\xi) - E_k(z,x)}{t} + \sum_{\alpha\in R^+} k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, x\rangle}(E_k(z,x) - E_k(z, r_\alpha x))$$

is given in terms of the Dunkl kernels, it belongs to $\mathscr{F}_k(\mathbb{C}^N)$. Clearly $\lim_{t\downarrow 0} \psi(t,z,x) = \langle \xi, z\rangle E_k(z,x)$. On the other hand

$$\begin{aligned}
&\|\psi(t,z,x) - \psi(t',z,x)\|_k^2 \\
&= \left\| \frac{E_k(z,x+t\xi) - E_k(z,x)}{t} \right\|_k^2 + \left\| \frac{E_k(z,x+t'\xi) - E_k(z,x)}{t'} \right\|_k^2 \\
&-2\mathrm{Rel} \left\langle\!\!\left\langle \frac{E_k(z,x+t\xi) - E_k(z,x)}{t}, \frac{E_k(z,x+t'\xi) - E_k(z,x)}{t'} \right\rangle\!\!\right\rangle_k \\
&= \left[\frac{E_k(x,x) + E_k(x+t\xi, x+t\xi) - 2E_k(x, x+t\xi)}{t^2} \right] + \left[\frac{E_k(x,x) + E_k(x+t'\xi, x+t'\xi)}{t'^2} \right. \\
&\left. -2\frac{E_k(x, x+t'\xi)}{t'^2} \right] - 2\left[\frac{E_k(x,x) + E_k(x+t\xi, x+t'\xi) - E_k(x, x+t\xi) - E_k(x+t'\xi)}{tt'} \right].
\end{aligned}$$

By the Taylor series (2.3) of $E_k(x,y)$, the asymptotic expansions of the above three terms between the brackets are

$$\langle \xi, x\rangle^2 E_k(x,x) + O(t), \quad \langle \xi, x\rangle^2 E_k(x,x) + O(t'), \quad \langle \xi, x\rangle^2 E_k(x,x) + O(tt') \quad \text{as } t, t' \to 0,$$

respectively. Therefore

$$\lim_{t,t'\downarrow 0} \|\psi(t,z,x) - \psi(t',z,x)\|_k = 0.$$

Hence $\psi(t,z,x)$ converges in norm and in pointwise topology to $\langle \xi, z\rangle E_k(z,x)$, with $\langle \alpha, x\rangle \neq 0$ for all $\alpha \in R$. In particular, $\langle \xi, z\rangle E_k(z,x)$ belongs to $\mathscr{F}_k(\mathbb{C}^N)$.

Fix $z_0 \in \mathbb{C}^N$ such that $\langle \alpha, z_0\rangle \neq 0$ for all $\alpha \in R$, and write $f_n(z) = \langle z, \xi\rangle E_k(z, \frac{z_0}{n})$. Next we will prove that $f_n(z)$ converges to $\langle z, \xi\rangle$ in $\mathscr{F}_k(\mathbb{C}^N)$ as $n \to \infty$. From the above discussion, it follows that $\{f_n\}_n \in \mathscr{F}_k(\mathbb{C}^N)$. Further $\{f_n(z)\}_n$ is convergent for all $z \in \mathbb{C}^N$. We claim that $\|f_n\|_k \leq M$ for some constant M and for all n. Therefore, by Lemma 3.4, we can deduce that $f_n(z)$ converges weakly to $\langle z, \xi\rangle \in \mathscr{F}_k(\mathbb{C}^N)$. To prove the claim, notice that

$$\begin{aligned}
\|f_n\|_k &\leq \|\partial_\xi E_k(\cdot, z_0/n)\|_k + \Big\| \sum_{\alpha\in R^+} k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, z_0/n\rangle}(E_k(\cdot, z_0/n) - E_k(\cdot, r_\alpha z_0/n)) \Big\|_k \\
&\leq \|\partial_\xi E_k(\cdot, z_0/n)\|_k + \left(2\sum_{\alpha\in R^+} k_\alpha^2 \frac{\langle \alpha, \xi\rangle^2}{\langle \alpha, z_0/n\rangle^2}(E_k(z_0/n, z_0/n) - E_k(z_0/n, r_\alpha z_0/n)) \right)^{1/2}.
\end{aligned}$$

Using the fact that

$$\frac{E_k(x,x)-E_k(x,r_\alpha x)}{\langle \alpha, x\rangle}=\int_0^1 \partial_\alpha E_k(x,x-y\langle\alpha,x\rangle\alpha)dy,$$

and [29] for a suitable growth estimate for $\partial_\alpha E_k$, we can deduce that $\|f_n\|_k \le M$.
For higher powers of z, one can reproduce the same argument.

To prove the density of $\mathscr{P}(\mathbb{C}^N)$, we need to introduce some notations. Let $\mathscr{P}_{\mathbb{R}}(\mathbb{R}^N)$ be the space of real-coefficients polynomials on $\mathbb{R}^N$, and, for $p, q \in \mathscr{P}(\mathbb{R}^N)$, set $[p,q]_k = p(T)q(0)$. Here $p(T)$ is the operator derived from $p(x)$ by replacing x_i by $T_i(k)$. Let $\{\varphi_{\mathbf{m}} | \mathbf{m} \in \mathbb{Z}_+^N\}$ be an orthonormal basis for $\mathscr{P}_{\mathbb{R}}(\mathbb{R}^N)$ with respect to $[\cdot,\cdot]_k$. By [31], the reproducing kernel $\mathbb{K}$ can be written as

$$\mathbb{K}(z,w)=\sum_{\mathbf{m}\in\mathbb{Z}_+^N}\varphi_{\mathbf{m}}(z)\overline{\varphi_{\mathbf{m}}(w)}, \qquad z,w\in\mathbb{C}^N,$$

where we extend $\varphi_{\mathbf{m}}$ to be in $\mathscr{P}_{\mathbb{R}}(\mathbb{C}^N)$. By Proposition 3.8 below, the inner product $\langle\langle\cdot,\cdot\rangle\rangle_k$ on $\mathscr{P}_{\mathbb{R}}(\mathbb{C}^N)$ coincides with the brackets $[\cdot,\cdot]_k$. Namely, $\{\varphi_{\mathbf{m}} | \mathbf{m} \in \mathbb{Z}_+^N\}$ now forms an orthonormal system in $\mathscr{P}(\mathbb{C}^N) \subset \mathscr{F}_k(\mathbb{C}^N)$ with respect to $\langle\langle\cdot,\cdot\rangle\rangle_k$. Thus, for fixed $w \in \mathbb{C}^N$, the sum

$$\sum_{\mathbf{m}\in\mathbb{Z}_+^N}\varphi_{\mathbf{m}}(\cdot)\overline{\varphi_{\mathbf{m}}(w)}$$

is convergent in $\mathscr{F}_k(\mathbb{C}^N)$, since $\sum_{\mathbf{m}\in\mathbb{Z}_+^N}|\varphi_{\mathbf{m}}(w)|^2 = \mathbb{K}(w,w) < \infty$. Its limit will be $\mathbb{K}_w(\cdot)$, since $\|\cdot\|_k$-convergence implies pointwise convergence. Hence, $\mathbb{K}_w \in \overline{\mathscr{P}(\mathbb{C}^N)}$ and $\mathscr{F}_k(\mathbb{C}^N) = \overline{\mathscr{P}(\mathbb{C}^N)}$. □

Note that the argument used in the proof above establishes also that $\{\varphi_{\mathbf{m}} | \mathbf{m} \in \mathbb{Z}_+^N\}$ forms an orthonormal basis for $\mathscr{F}_k(\mathbb{C}^N)$.

For $\xi \in \mathbb{C}^N$, denote by M_ξ the operator defined for $f \in \mathscr{F}_k(\mathbb{C}^N)$ by $M_\xi(f)(z) = \langle z, \xi\rangle f(z)$. Further, we set

$$\mathbb{D}(M_\xi)=\left\{f\in\mathscr{F}_k(\mathbb{C}^N) \mid M_\xi(f)\in\mathscr{F}_k(\mathbb{C}^N)\right\},$$
$$\mathbb{D}(T_\xi(k))=\left\{f\in\mathscr{F}_k(\mathbb{C}^N) \mid T_\xi(k)(f)\in\mathscr{F}_k(\mathbb{C}^N)\right\}.$$

Theorem 3.7. *The operators* M_ξ *and* $T_\xi(k)$ *are closed, densely defined operators on* $\mathscr{F}_k(\mathbb{C}^N)$ *such that* $T_\xi(k)$ *is the adjoint operator of* M_ξ, *and* $\mathbb{D}(T_\xi(k)) = \mathbb{D}(M_\xi)$.

Proof. Clearly the operators $T_\xi(k)$ and M_ξ are densely defined (the set of polynomials is contained in each of their domains). Let $(f_n, T_\xi(k)f_n)$ be a sequence in the graph of $T_\xi(k)$, and assume that $(f_n, T_\xi(k)f_n) \to (f, g) \in \mathscr{F}_k(\mathbb{C}^N)\times\mathscr{F}_k(\mathbb{C}^N)$. Now $\lim_{n\to\infty}\|f_n\|_k = \|f\|_k$ and $\lim_{n\to\infty}\|T_\xi(k)f_n\|_k = \|g\|_k$. Since strong convergence implies pointwise convergence, therefore, for all $z \in \mathbb{C}^N$, $f(z) = \lim_{n\to\infty} f_n(z)$ and $g(z) = \lim_{n\to\infty} T_\xi(k)f_n(z)$. Notice that $\{f_n(z)\}$ converges locally uniformly to $f(z)$, and therefore $g(z) = \lim_{n\to\infty} T_\xi(k)f_n(z) = T_\xi(k)f(z)$.

Using Corollary 3.5, we can deduce that $T_\xi(k)$ is closed. The same argument can be used for the operator M_ξ.

Recall that $\mathbb{S}_k(\mathbb{C}^N) = \langle \mathbb{K}_z \mid z \in \mathbb{C}^N \rangle$ is dense in $\mathscr{F}_k(\mathbb{C}^N) = \overline{\mathbb{S}_k(\mathbb{C}^N)}$. By the definition and properties of $\mathbb{K}$, it follows that

$$\begin{aligned} \langle\!\langle T_\xi(k)\mathbb{K}_z, \mathbb{K}_w \rangle\!\rangle_k = T_\xi(k)\mathbb{K}(w,z) &= \overline{\langle z, \xi\rangle}\mathbb{K}(w,z) \\ &= \overline{\langle z, \xi\rangle \mathbb{K}(z,w)} = \overline{\langle\!\langle M_\xi \mathbb{K}_w, \mathbb{K}_z \rangle\!\rangle_k} = \langle\!\langle \mathbb{K}_z, M_\xi \mathbb{K}_w \rangle\!\rangle_k. \end{aligned}$$

Therefore, $\langle\!\langle T_\xi(k)f, g\rangle\!\rangle_k = \langle\!\langle f, M_\xi g\rangle\!\rangle_k$ for all $f, g \in \mathbb{S}_k(\mathbb{C}^N)$. Denote by $T_\xi^*(k)$ and M_ξ^* the adjoint operators of $T_\xi(k)$ and M_ξ, respectively. Hence, $\langle f, T_\xi^*(k)g\rangle = \langle f, M_\xi g\rangle$ for $f, g \in \mathbb{S}_k(\mathbb{C}^N)$. Since $\mathbb{S}_k(\mathbb{C}^N)$ is total in $\mathscr{F}_k(\mathbb{C}^N)$, then one can extend the equality for $f \in \mathscr{F}_k(\mathbb{C}^N)$. We can use the same argument for $g \in \mathbb{S}_k(\mathbb{C}^N)$ with $\langle T_\xi^{**}(k)f, g\rangle = \langle M_\xi^* f, g\rangle$. Hence $\langle T_\xi^{**}(k)f, g\rangle = \langle M_\xi^* f, g\rangle$ for $f, g \in \mathscr{F}_k(\mathbb{C}^N)$, whenever the both sides make sense. As $T_\xi(k)$ is closed, it follows that $T_\xi(k) = T_\xi^{**}(k) = M_\xi^*$.

For the last assertion, let $f \in \mathscr{F}_k(\mathbb{C}^N)$. Using the fact that

$$(1 - r_\alpha)\{\langle \eta, z\rangle f(z)\} = \langle \eta, z\rangle \left(f(z) - f(r_\alpha z)\right) + \left(\langle \eta, z\rangle - \langle \eta, r_\alpha z\rangle\right) f(r_\alpha z),$$

and

$$\langle \eta, z\rangle - \langle \eta, r_\alpha z\rangle = \langle \alpha, z\rangle\langle \alpha, \eta\rangle,$$

(recall our normalization $|\alpha|^2 = 2$), we obtain

$$\begin{aligned} & T_\xi(k)\{\langle \eta, z\rangle f(z)\} \\ = \; & \langle \eta, \xi\rangle f(z) + \langle \eta, z\rangle \left\{ \partial_\xi f(z) + \sum_{\alpha \in R^+} k_\alpha \frac{\langle \alpha, \xi\rangle}{\langle \alpha, z\rangle}\left(f(z) - f(r_\alpha z)\right) \right\} \\ & + \sum_{\alpha \in R^+} k_\alpha \langle \alpha, \xi\rangle\langle \alpha, \eta\rangle f(r_\alpha z) \\ = \; & \langle \eta, \xi\rangle f(z) + \langle \eta, z\rangle T_\xi(k) f(z) + \sum_{\alpha \in R^+} k_\alpha \langle \alpha, \xi\rangle\langle \alpha, \eta\rangle f(r_\alpha z). \end{aligned} \tag{3.2}$$

Thus, for instance if $p \in \mathscr{P}(\mathbb{C}^N)$ (or in the respective domains), and $\xi \in \mathbb{C}^N$ with $|\xi| = 1$, we have

$$\|M_\xi p\|_k^2 = \|p\|_k^2 + \|T_\xi(k)p\|_k^2 + \sum_{\alpha \in R^+} k_\alpha \langle \alpha, \xi\rangle^2 \langle\!\langle r_\alpha \cdot p, p\rangle\!\rangle_k. \tag{3.3}$$

Now, let $f \in \mathbb{D}(T_\xi(k))$, i.e. $f \in \mathscr{F}_k(\mathbb{C}^N)$ such that $T_\xi(k)f \in \mathscr{F}_k(\mathbb{C}^N)$. Since $\mathscr{P}(\mathbb{C}^N)(\subset \mathbb{D}(T_\xi(k)))$ is dense in $\mathscr{F}_k(\mathbb{C}^N)$, and the graph of $T_\xi(k)$ is closed, assume $(p_n, T_\xi(k)p_n)_n$, with $(p_n)_n \in \mathscr{P}(\mathbb{C}^N)$, converges to $(f, T_\xi(k)f)$ as $n \to \infty$ [27, Proposition VIII.1.1]. By (3.3) it follows that $(M_\xi p_n)_n$ is a Cauchy sequence. Therefore $(M_\xi p_n)_n$ converges in $\mathscr{F}_k(\mathbb{C}^N)$, and, by the closedness of M_ξ, to $M_\xi f$. The same argument can be employed to prove the converse direction. Hence $\mathbb{D}(T_\xi(k)) = \mathbb{D}(M_\xi)$. $\square$

Due to the commutativity of the Dunkl operators $T_\xi(k)$, the linear map $\xi \mapsto T_\xi(k)$ can be extended in a unique way to an algebra homomorphism from the symmetric algebra $S(\mathbb{C}^N)$ to $\mathrm{End}(\mathbb{C}^N)$. The image of $p \in S(\mathbb{C}^N)$ will be denoted by $p(T)$, where $p(T)$ is the operator formed by replacing z_i by $T_i(k)$ for $1 \leq i \leq N$.

Proposition 3.8. *For* $p, q \in \mathscr{P}(\mathbb{C}^N)$, *the inner product* $\langle\langle \cdot, \cdot \rangle\rangle_k$ *satisfies*

$$\langle\langle p, q \rangle\rangle_k = p(T)\overline{q}(z)\big|_{z=0},$$

where $\bar{q}$ *is the polynomial defined by* $\bar{q}(z) = \overline{q(\bar{z})}$.

Proof. Recall that, for $\xi \in \mathbb{C}^N$, the two operators M_ξ and $T_\xi(k)$ are densely defined, and the set of polynomials is contained in their domains. For $p, q \in \mathscr{P}(\mathbb{C}^N)$, set

$$\lll p, q \ggg_k := p(T)\bar{q}(z)_{|z=0}.$$

By the commutativity of the Dunkl operators, clearly

$$\lll M_\xi p, q \ggg_k = \lll p, T_\xi(k) q \ggg_k .$$

Denote by $\mathscr{P}^{(n)}(\mathbb{C}^N)$ the set of polynomials of total degree less than or equal to n. First, notice that

$$\langle\langle 1, 1 \rangle\rangle_k = \langle\langle \mathbb{K}_0, \mathbb{K}_0 \rangle\rangle_k = \mathbb{K}(0,0) = 1 = \lll 1, 1 \ggg_k .$$

Therefore the statement holds for constant polynomials. Moreover

$$\langle\langle 1, \langle z, \xi \rangle \rangle\rangle_k = \langle\langle T_\xi(k) 1, 1 \rangle\rangle_k = 0,$$

where, on the other hand, we have

$$\lll 1, \langle z, \xi \rangle \ggg_k = \overline{\langle \bar{z}, \xi \rangle}_{|z=0} = 0.$$

It is now easy to check that the statement holds if p or q is constant. To prove the statement in general, we will use the induction on $\max(\text{total deg}(p), \text{total deg}(q))$ for $p, q \in \mathscr{P}^{(n)}(\mathbb{C}^N)$. Assume the statement holds for $n-1$ in place of n. For $p \in \mathscr{P}^{(n-1)}(\mathbb{C}^N)$ and $q \in \mathscr{P}^{(n)}(\mathbb{C}^N)$, we have

$$\langle\langle M_\xi p, q \rangle\rangle_k = \langle\langle p, T_\xi(k) q \rangle\rangle_k.$$

Since $T_\xi(k)$ is homogeneous of degree -1 on $\mathscr{P}(\mathbb{C}^N)$, it follows, by the inductive hypothesis, that

$$\langle\langle p, T_\xi(k) q \rangle\rangle_k = \lll p, T_\xi(k) q \ggg_k = \lll M_\xi p, q \ggg_k,$$

which leads to

$$\langle\langle M_\xi p, q \rangle\rangle_k = \lll M_\xi p, q \ggg_k .$$

This finishes the proof. $\square$

Below, we collect some fundamental properties of the inner product $\langle\langle \cdot,\cdot\rangle\rangle_k$ on $\mathscr{P}(\mathbb{C}^N)$. For $n \in \mathbb{Z}^+$, let $\mathscr{P}_n$ be the space of homogeneous polynomials in $\mathscr{P}(\mathbb{C}^N)$ of degree n.

Lemma 3.9. *For non-negative multiplicity functions* k

(i) $\langle\langle \mathscr{P}_n, \mathscr{P}_m\rangle\rangle_k = 0$ *if* $n \neq m$; *and*

(ii) $\langle\langle V_k p, q\rangle\rangle_k = \langle\langle p, q\rangle\rangle_0$ *for* $p, q \in \mathscr{P}(\mathbb{C}^N)$.

Proof. (i) The assertion follows from Proposition 3.8.
(ii) By statement (i), it is enough to consider $p, q \in \mathscr{P}_n$. Using the fact that $T_\xi(k)\circ V_k = V_k\circ\partial_\xi$, and the fact that $V_k|_{\mathscr{P}_0} = \mathrm{id}$, we obtain

$$\begin{aligned}\langle\langle V_k p, q\rangle\rangle_k = \overline{\langle\langle q, V_k p\rangle\rangle_k} &= \overline{q(T_\xi)(V_k\overline{p})(0)}\\ &= \overline{V_k(q(\partial_\xi)(\overline{p})(0))} = \overline{q(\partial_\xi)(\overline{p})(0)} = \overline{\langle\langle q, p\rangle\rangle_0} = \langle\langle p, q\rangle\rangle_0.\end{aligned}$$

□

We conclude this section by studying the possibility of seeing the norm in the subspace $\mathscr{F}_k(\mathbb{C}^N)^G$ of G-invariant functions in $\mathscr{F}_k(\mathbb{C}^N)$, as an $\mathscr{L}^2$-norm.

Let $\mathscr{F}_k(\mathbb{C}^N)^G$ be the Hilbert space of G-invariant functions in $\mathscr{F}_k(\mathbb{C}^N)$, and fix an orthonormal basis $\{\xi_1, \ldots, \xi_N\}$ of $\mathbb{C}^N$. The G-equivariance of the Dunkl operators implies that $\Delta_k = \sum_{i=1}^N T_{\xi_i}(k)^2$ is G-equivariant, i.e. $g\circ\Delta\circ g^{-1} = \Delta_k$. Set $M^2 := \sum_{i=1}^N M_{\xi_i}^2$, and define the weight function $\tilde{w}_k(z) := w_k(z)\overline{w_k(z)}$. Let $\rho_k(x,y)$, for $x, y \in \mathbb{R}^N$, be a positive real function which assumed to define the inner product in $\mathscr{F}_k(\mathbb{C}^N)^G$ by

$$\langle\langle f, g\rangle\rangle_k = \int_{\mathbb{C}^N} f(z)\overline{g(z)}\rho_k(x,y)\tilde{w}_k(z)dz, \qquad f, g \in \mathscr{F}_k(\mathbb{C}^N)^G,$$

such that

$$\langle\langle \Delta_k f, g\rangle\rangle_k = \langle\langle f, M^2 g\rangle\rangle_k, \qquad f, g \in \mathscr{F}_k(\mathbb{C}^N)^G. \tag{3.4}$$

Here dz is the $2N$-dimensional volume element $\prod dx_i \prod dy_i$. Notice that condition (3.4) holds for all elements in $\mathscr{F}_k(\mathbb{C}^N)$, whenever both sides of the equation make sense. The function ρ_k should also satisfy an exponential decay at infinity (recall that $\mathscr{P}(\mathbb{C}^N)^G \subset \mathscr{F}_k(\mathbb{C}^N)^G$).

A short calculation shows that

$$\int_{\mathbb{C}^N}\Big[T_\xi(k)f(z)\Big]g(z)\tilde{w}_k(z)dz = -\int_{\mathbb{C}^N} f(z)\Big[T_\xi(k)g(z)\Big]\tilde{w}_k(z)dz,$$

for all suitably decaying functions f and g. Therefore

$$\int_{\mathbb{C}^N}\Big[\Delta_k f(z)\Big]g(z)\tilde{w}_k(z)dz = \int_{\mathbb{C}^N} f(z)\Big[\Delta_k g(z)\Big]\tilde{w}_k(z)dz.$$

Thus (3.4) becomes a condition on ρ_k, namely

$$\int_{\mathbb{C}^N} f(z)\Delta_k\Big[\overline{g(z)}\rho_k(z)\Big]\tilde{w}_k(z)dz = \int_{\mathbb{C}^N} f(z)|\overline{z}|^2\overline{g(z)}\rho_k(z)\tilde{w}_k(z)dz, \tag{3.5}$$

with $|\bar{z}|^2 = \sum_{i=1}^N \bar{z}_i^2$ for $z \in \mathbb{C}^N$. Since g is a G-invariant holomorphic function, then we may rewrite (3.5) as

$$\int_{\mathbb{C}^N} f(z)\overline{g(z)}\Big[\Delta_k\rho_k(z)\Big]\tilde{w}_k(z)dz = \int_{\mathbb{C}^N} f(z)\overline{g(z)}|\bar{z}|^2\rho_k(z)\tilde{w}_k(z)dz, \qquad f, g \in \mathscr{F}_k(\mathbb{C}^N)^G.$$

Therefore, the function ρ_k satisfies

$$p(T(k))\rho_k(z) = p(-\bar{z})\rho_k(z), \qquad \forall p \in S(\mathbb{C}^N)^G. \tag{3.6}$$

Unfortunately, the natural solution of the differential system (3.6) which is $E_k(z, -\bar{z})$, or even the Bessel function $J_G(z, -\bar{z}) = \sum_{g\in G} E_k(gz, -\bar{z})$, cannot be a candidate for the function ρ_k as the rank one case shows by looking at the asymptotic growth of $E_k(z, -\bar{z}) = e^{-\|z\|^2}{}_1F_1(k; 2k+1; 2\|z\|^2)$ as $\|z\| \to \infty$ (here $\|z\|^2 = z\bar{z}$). In [5] Cholewinski studied the case of even functions in $\mathscr{F}_k(\mathbb{C})$, where he used $\rho_k(z) = \|z\|^{1-2k}\mathscr{K}_{k-1/2}(\|z\|^2)$, with $z \in \mathbb{C}$ and $\mathscr{K}_\nu$ is the Bessel function of the third kind. See Example 4.14 below, for a complete investigation on the measure associated with $\mathscr{F}_k(\mathbb{C})$ by using Cholewinski's result.

We conjecture that there exists a Bessel function $\mathbf{K}(k, \cdot, \cdot)$ on $\mathbb{C}^N \times \mathbb{C}^N$, with exponential decay at infinity, such that

$$p(T_\xi^z(k))\mathbf{K}_k(z, w) = p(-w)\mathbf{K}_k(z, w), \qquad \forall p \in S(\mathbb{C}^N)^G. \tag{3.7}$$

Here the superscript z denotes that the operators act with respect to the z-variable. The definition and properties of $\mathbf{K}(k, \cdot, \cdot)$ extend naturally those of its classical counterpart $\mathbf{K}(k, z, w) = (zw)^{1/2-k}\mathscr{K}_{k-1/2}(zw)$ for $z, w \in \mathbb{C}$. In the general setting, $\mathbf{K}(k, \cdot, \cdot)$ deserves the name of Bessel function of type three. A closer investigation of such generalized Bessel functions of type three will appear in a forthcoming paper.

4. The Segal-Bargmann transform associated with Coxeter groups

In this section we give a generalized Segal-Bargmann transform between $\mathscr{L}^2(\mathbb{R}^N, w_k)$ and the Fock space $\mathscr{F}_k(\mathbb{C}^N)$ via a restriction principle, i.e. polarization of suitable restriction map. This idea of restriction was first applied to the Weyl transform in [22], and later to the Segal-Bargmann transform associated with weighted Bergman spaces on bounded symmetric domains (cf. [23, 7, 34]).

For $t > 0$ and $z, w \in \mathbb{C}^N$, set

$$\Gamma_k(t, z, w) = \frac{1}{(2t)^{\gamma+N/2}c_k} e^{-(|z|^2+|w|^2)/4t} E_k\Big(\frac{z}{\sqrt{2t}}, \frac{w}{\sqrt{2t}}\Big).$$

The kernel $\Gamma_k(t, z, w)$ was introduced in [29] as a generalized heat kernel. Recall that $|z|^2 = \sum_{i=1}^N z_i^2$ for $z \in \mathbb{C}^N$.

Let $\mathscr{L}^2(\mathbb{R}^N, w_k)$ be the space of $\mathscr{L}^2$-functions on $\mathbb{R}^N$ with respect to the weight function w_k. The notation $\|\cdot\|$ will be set for the norm in $\mathscr{L}^2(\mathbb{R}^N, w_k)$.

Let $\mathscr{R}_k$ be the restriction map $\mathscr{R}_k : \mathscr{F}_k(\mathbb{C}^N) \to \mathscr{L}^2(\mathbb{R}^N, w_k)$, given by

$$\mathscr{R}_k f(x) = e^{-|x|^2/2} f(x), \qquad x \in \mathbb{R}^N.$$

The map $\mathscr{R}_k$ is a closed, densely defined operator from $\mathscr{F}_k(\mathbb{C}^N)$ into $\mathscr{L}^2(\mathbb{R}^N, w_k)$ with dense image (see for instance [29, Theorem 3.2]). We should notice here that our normalization of $\mathscr{L}^2(\mathbb{R}^N, w_k)$ is slightly different from the one in [29].

Now consider the adjoint $\mathscr{R}_k^* : \mathscr{L}^2(\mathbb{R}^N, w_k) \to \mathscr{F}_k(\mathbb{C}^N)$ as a densely defined operator.

Proposition 4.1. *For* $f \in \mathscr{L}^2(\mathbb{R}^N, w_k)$, *the integral*

$$\mathscr{R}_k \mathscr{R}_k^* f(y) = c_k \int_{\mathbb{R}^N} f(x) \Gamma_k(\frac{1}{2}, x, y) w_k(x) dx$$

converges absolutely for a.e. $y \in \mathbb{R}^N$. *The function* $\mathscr{R}_k \mathscr{R}_k^* f$ *thus defined is in* $\mathscr{L}^2(\mathbb{R}^N, w_k)$ *and* $\|\mathscr{R}_k \mathscr{R}_k^*\| \leq c_k$.

Proof. Since $\mathbb{K}$ is the reproducing kernel of $\mathscr{F}_k(\mathbb{C}^N)$, then for $f \in \mathscr{L}^2(\mathbb{R}^N, w_k)$ and $z \in \mathbb{C}^N$

$$\begin{aligned}
\mathscr{R}_k^* f(z) &= \langle\langle \mathscr{R}_k^* f, \mathbb{K}_z \rangle\rangle_k \\
&= (f, \mathscr{R}_k \mathbb{K}_z)_{\mathscr{L}^2} \\
&= \int_{\mathbb{R}^N} f(x) \mathscr{R}_k E_k(z, x) w_k(x) dx \\
&= \int_{\mathbb{R}^N} f(x) E_k(z, x) e^{-|x|^2/2} w_k(x) dx.
\end{aligned}$$

Therefore, for $y \in \mathbb{R}^N$

$$\begin{aligned}
\mathscr{R}_k \mathscr{R}_k^* f(y) &= e^{-|y|^2/2} \mathscr{R}_k^* f(y) \\
&= \int_{\mathbb{R}^N} f(x) e^{-(|x|^2+|y|^2)/2} E_k(x, y) w_k(x) dx \\
&= c_k \int_{\mathbb{R}^N} f(x) \Gamma_k(\frac{1}{2}, x, y) w_k(x) dx.
\end{aligned}$$

By Hölder's inequality we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |f(x) \Gamma(\frac{1}{2}, x, y)| w_k(x) dx &\leq \left[\int_{\mathbb{R}^N} \Gamma(\frac{1}{2}, x, y) w_k(x) dx \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^N} \Gamma(\frac{1}{2}, x, y) |f(x)|^2 w_k(x) dx \right]^{\frac{1}{2}} \\
&= \left[\int_{\mathbb{R}^N} \Gamma(\frac{1}{2}, x, y) |f(x)|^2 w_k(x) dx \right]^{\frac{1}{2}}
\end{aligned}$$

for a.e. $y \in \mathbb{R}^N$. Above we used the fact that $\Gamma_k(t,x,y) > 0$ and $\int_{\mathbb{R}^N} \Gamma_k(t,x,y) w_k(x)dx = 1$ for all $t > 0$ (cf. [30]). Hence, by Tonelli's theorem

$$\begin{aligned}
\|\mathscr{R}_k\mathscr{R}_k^* f\|^2 &\leq c_k^2 \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} |f(x)\Gamma(\frac{1}{2},x,y)| w_k(x)dx\right]^2 w_k(y)dy \\
&\leq c_k^2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Gamma(\frac{1}{2},x,y)|f(x)|^2 w_k(x)w_k(y)dxdy \\
&= c_k^2 \int_{\mathbb{R}^N} |f(x)|^2 w_k(x)dx \\
&= c_k^2\|f\|^2.
\end{aligned}$$

Since $f \in \mathscr{L}^2(\mathbb{R}^N, w_k)$, $\mathscr{R}_k\mathscr{R}_k^* f$ is well defined a.e., and $\|\mathscr{R}_k\mathscr{R}_k^* f\| \leq c_k\|f\|$. □

Using the integral formula (2.5), the following observation holds. For the case of flat symmetric space of type C_N or D_N ($N \geq 3$) and slightly different transformation, this observation was made in [34].

Theorem 4.2. *For $x \in \mathbb{R}^N$ and $z \in \mathbb{C}^N$, the transform $\mathscr{R}_k\mathscr{R}_k^*$ has $E_k(x,z)$ as an eigenfunction. More precisely*

$$\mathscr{R}_k\mathscr{R}_k^* E_k(x,z) = c_k e^{|z|^2/2} E_k(x,z).$$

By Proposition 4.1, the operator $\mathscr{R}_k\mathscr{R}_k^*$ is well defined and by definition is positive. We can therefore define $\sqrt{\mathscr{R}_k\mathscr{R}_k^*}$. Thus there exists an isometry $\mathscr{B}_k$ so that $\mathscr{R}_k^* = \mathscr{B}_k\sqrt{\mathscr{R}_k\mathscr{R}_k^*}$. Since $\mathscr{R}_k = \sqrt{\mathscr{R}_k\mathscr{R}_k^*}\mathscr{B}_k^*$ and Image$(\mathscr{R}_k)$ is dense, it follows that $\mathscr{B}_k$ is a unitary isomorphism. In considering $\mathscr{B}_k f(a+ib)$, one may interpret a as a position variable and b as a frequency variable. In the context of quantum mechanics, the frequency variable has the interpretation of momentum.

Theorem 4.3. *The unitary isomorphism $\mathscr{B}_k : \mathscr{L}^2(\mathbb{R}^N, w_k) \to \mathscr{F}_k(\mathbb{C}^N)$ is given by*

$$\mathscr{B}_k f(z) = 2^{\gamma+N/2} c_k^{-1/2} e^{-|z|^2/2} \int_{\mathbb{R}^N} f(x) E_k(\sqrt{2}x, \sqrt{2}z) e^{-|x|^2} w_k(x)dx.$$

The transformation $\mathscr{B}_k$ is called the generalized Segal-Bargmann transform associated with G.

Proof. Let $f \in \mathscr{L}^2(\mathbb{R}^N, w_k)$. Since

$$\mathscr{R}_k\mathscr{R}_k^* f(y) = c_k \int_{\mathbb{R}^N} f(x)\Gamma_k(\frac{1}{2},x,y) w_k(x)dx,$$

it follows, using again [30] and mainely the positivity of the heat kernel as an operator, that

$$|\mathscr{R}_k| f(y) := \sqrt{\mathscr{R}_k\mathscr{R}_k^*} f(y) = \sqrt{c_k}\int_{\mathbb{R}^N} f(x)\Gamma_k(\frac{1}{4},x,y) w_k(x)dx.$$

Using the integral representation of $\mathscr{R}_k^*$, given in the proof of Proposition 4.1, we obtain

$$\begin{aligned}
\mathscr{R}_k^* f(z) &= \int_{\mathbb{R}^N} f(x) E_k(x,z) e^{-|x|^2/2} w_k(x)\,dx \\
&= c_k e^{|z|^2/2} \int_{\mathbb{R}^N} f(x) \Gamma_k(\frac{1}{2}, x, z) w_k(x)\,dx \\
&= c_k e^{|z|^2/2} \int_{\mathbb{R}^N} f(x) \left[\int_{\mathbb{R}^N} \Gamma_k(\frac{1}{4}, x, y) \Gamma_k(\frac{1}{4}, z, y) w_k(y)\,dy \right] w_k(x)\,dx \\
&= c_k e^{|z|^2/2} \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} f(x) \Gamma_k(\frac{1}{4}, x, y) w_k(x)\,dx \right] \Gamma_k(\frac{1}{4}, y, z) w_k(y)\,dy \\
&= \sqrt{c_k} e^{|z|^2/2} \int_{\mathbb{R}^N} |\mathscr{R}_k| f(y) \Gamma_k(\frac{1}{4}, y, z) w_k(y)\,dy \\
&= \mathscr{B}_k(|\mathscr{R}_k| f)(z).
\end{aligned}$$

To obtain the above third equality, we used the fact that

$$\begin{aligned}
\int_{\mathbb{R}^N} \Gamma_k(\frac{1}{4}, x, y) \Gamma_k(\frac{1}{4}, z, y) w_k(y)\,dy &= \frac{2^{2\gamma+N}}{c_k^2} e^{-|x|^2-|z|^2} \int_{\mathbb{R}^N} E_k(x, 2y) E_k(z, 2y) e^{-2|y|^2} w_k(y)\,dy \\
&= c_k^{-1} e^{-(|x|^2+|z|^2)/2} E_k(x,z) \\
&= \Gamma_k(\frac{1}{2}, x, z).
\end{aligned}$$

□

Remark 4.4. For the special case $k \equiv 0$

$$\mathscr{B}_0 f(z) = (2/\pi)^{N/4} \int_{\mathbb{R}^N} e^{-|x|^2 + 2\langle x, z\rangle - |z|^2/2} f(x)\,dx.$$

This compares well with the classical Segal-Bargmann transform (cf. [16, pp. 40]).

For $p, q \in \mathscr{P}(\mathbb{R}^N)$, set $[p,q]_k = p(T)q(0)$. Here $p(T)$ is the operator derived from $p(x)$ by replacing x_i by $T_i(k)$. Due to Dunkl [11], the pairing $[\cdot,\cdot]_k$ is in fact a scalar product on the $\mathbb{R}$-vector space $\mathscr{P}_{\mathbb{R}}(\mathbb{R}^N)$ of real valued polynomials on $\mathbb{R}^N$.

Recall that $\{\varphi_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{Z}_+^N\}$ forms an orthonormal basis for $\mathscr{P}_{\mathbb{R}}(\mathbb{R}^N)$ with respect to $[\cdot,\cdot]_k$, such that $\varphi_{\mathbf{m}} \in \mathscr{P}_{|\mathbf{m}|}$. Here $|\mathbf{m}| = m_1 + \cdots + m_N$. For instance, if $k \equiv 0$, the natural choice of the basis $\{\varphi_{\mathbf{m}}\}$ is $\varphi_{\mathbf{m}}(x) = x^{\mathbf{m}}/\sqrt{\mathbf{m}!}$.

In [30], Rösler defined generalized Hermite polynomials $\{ H_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{Z}_+^N\}$ and Hermite functions $\{ h_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{Z}_+^N\}$, associated with the basis $\{\varphi_{\mathbf{m}}\}$, by

$$H_{\mathbf{m}}(x) = e^{-\Delta_k/2} \varphi_{\mathbf{m}}(x), \qquad h_{\mathbf{m}}(x) = e^{-|x|^2/4} H_{\mathbf{m}}(x), \qquad x \in \mathbb{R}^N.$$

For $\mathbf{m} \in \mathbb{Z}_+^N$, set $\Psi_{\mathbf{m}} := 2^{\gamma+N/2} c_k^{-1/2} d_2 \circ h_{\mathbf{m}}$, where d_2 denotes the dilation operator on functions by 2, i.e. $d_2 f(x) = f(2x)$.

Proposition 4.5. *For $z \in \mathbb{C}^N$*

$$\mathscr{B}_k(\Psi_{\mathbf{m}})(z) = \varphi_{\mathbf{m}}(z).$$

Since the set $\{\varphi_{\mathbf{m}} | \mathbf{m} \in \mathbb{Z}_+^N\}$ forms an orthonormal basis for $\mathscr{F}_k(\mathbb{C}^N)$ (see Section 3), it follows from Theorem 4.3 that the set $\{\Psi_{\mathbf{m}} | \mathbf{m} \in \mathbb{Z}_+^N\}$ forms an orthonormal basis for $\mathscr{L}^2(\mathbb{R}^N, w_k)$.

Proof. It is a well known fact that $E_k(\lambda z, w) = E_k(z, \lambda w)$ for all $\lambda \in \mathbb{C}$. Using the fact that $h_{\mathbf{m}}(x) = e^{-|x|^2/4} e^{-\Delta_k/2} \varphi_{\mathbf{m}}(x)$, and the integral formula (2.4), we obtain

$$\begin{aligned}
\mathscr{B}_k \Psi_{\mathbf{m}}(z) &= \sqrt{c_k} e^{|z|^2/2} \int_{\mathbb{R}^N} \Psi_{\mathbf{m}}(x) \Gamma_k(\frac{1}{4}, x, z) w_k(x) dx \\
&= c_k{}^{-1/2} 2^{\gamma+N/2} e^{-|z|^2/2} \int_{\mathbb{R}^N} \Psi_{\mathbf{m}}(x) e^{-|x|^2} E_k(\sqrt{2}x, \sqrt{2}z) w_k(x) dx \\
&= 2^{2\gamma+N} c_k^{-1} e^{-|z|^2/2} \int_{\mathbb{R}^N} h_{\mathbf{m}}(2x) E_k(\sqrt{2}x, \sqrt{2}z) e^{-|x|^2} w_k(x) dx \\
&= c_k^{-1} e^{-|z|^2/2} \int_{\mathbb{R}^N} h_{\mathbf{m}}(x) e^{-|x|^2/4} E_k(x, z) w_k(x) dx \\
&= c_k^{-1} e^{-|z|^2/2} \int_{\mathbb{R}^N} e^{-\Delta_k/2} \varphi_{\mathbf{m}}(x) E_k(x, z) e^{-|x|^2/2} w_k(x) dx \\
&= \varphi_{\mathbf{m}}(z).
\end{aligned}$$

Since the Segal-Bargmann transform $\mathscr{B}_k$ is a unitary isomorphism from $\mathscr{L}^2(\mathbb{R}^N, w_k)$ to $\mathscr{F}_k(\mathbb{C}^N)$, therefore the proposition holds. ☐

Remark 4.6.

(i) From the above proposition, it follows that

$$\mathscr{B}_k^{-1} = 2^{\gamma+N/2} c_k^{-1/2} e^{-|x|^2} d_2 \circ e^{-\Delta_k/2}.$$

(ii) Recall that the integral kernel $\mathscr{B}_k$ is a unitary isomorphism from $\mathscr{L}^2(\mathbb{R}^N, w_k)$ to $\mathscr{F}_k(\mathbb{C}^N)$ with kernel $2^{\gamma+N/2} c_k^{-1/2} e^{-|z|^2/2} e^{-|x|^2} E_k(\sqrt{2}z, \sqrt{2}x)$. As an immediate consequence of the above proposition, together with the unitarity of $\mathscr{B}_k$, the following generating relation holds

$$2^{\gamma+N/2} c_k^{-1/2} e^{-|z|^2/2} e^{-|x|^2} E_k(\sqrt{2}z, \sqrt{2}x) = \sum_{\mathbf{m} \in \mathbb{Z}_+^N} \varphi_{\mathbf{m}}(z) 2^{\gamma+N/2} c_k^{-1/2} h_{\mathbf{m}}(2x),$$

i.e.

$$e^{-|z|^2/2} E_k(z, x) = \sum_{\mathbf{m} \in \mathbb{Z}_+^N} \varphi_{\mathbf{m}}(z) H_{\mathbf{m}}(x). \tag{4.1}$$

This relation was also proved earlier in [30] by using a different approach.

The Dunkl transform, which shares many properties with the classical Fourier transform, was introduced in [12] and further studied in [21]. For our convenience, we will write the Dunkl transform as

$$\mathscr{D}_k f(\xi) = c_k^{-1} 2^{-\gamma - N/2} \int_{\mathbb{R}^N} f(x/2) E_k(-i\xi, x) w_k(x)\, dx, \qquad \xi \in \mathbb{R}^N.$$

Theorem 4.7. *The following diagram commutes*

$$\begin{array}{ccc} \mathscr{L}^2(\mathbb{R}^N, w_k) & \xrightarrow{\mathscr{B}_k} & \mathscr{F}_k(\mathbb{C}^N) \\ {\scriptstyle \mathscr{D}_k}\big\downarrow & & \big\downarrow{\scriptstyle (-i)^*} \\ \mathscr{L}^2(\mathbb{R}^N, w_k) & \xrightarrow{\mathscr{B}_k} & \mathscr{F}_k(\mathbb{C}^N) \end{array}$$

where $(-i)^* f(z) := f(-iz)$ *for* $f \in \mathscr{F}_k(\mathbb{C}^N)$.

Proof. For abbreviation write $\tilde{c} = c_k^{-1} 2^{-\gamma - N/2}$ and $\tilde{\tilde{c}} = c_k^{-1/2} 2^{\gamma + N/2}$. For $f \in \mathscr{L}^2(\mathbb{R}^N, w_k)$, we have

$$\begin{aligned} \mathscr{B}_k \mathscr{D}_k f(z) &= \tilde{\tilde{c}} e^{-|z|^2/2} \int_{\mathbb{R}^N} \mathscr{D}_k f(\xi) e^{-|\xi|^2} E_k(\sqrt{2}z, \sqrt{2}\xi) w_k(\xi)\, d\xi \\ &= \tilde{c}\tilde{\tilde{c}} e^{-|z|^2/2} \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} f(x/2) E_k(-i\xi, x) w_k(x)\, dx \right] e^{-|\xi|^2} E_k(\sqrt{2}z, \sqrt{2}\xi) w_k(\xi)\, d\xi \\ &= \tilde{c}\tilde{\tilde{c}} e^{-|z|^2/2} \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} E_k(-i\xi, x) E_k(\sqrt{2}z, \sqrt{2}\xi) e^{-|\xi|^2} w_k(\xi)\, d\xi \right] f(x/2) w_k(x)\, dx \\ &= \tilde{c}\tilde{\tilde{c}} c_k e^{-|z|^2/2} 2^{-\gamma - N/2} \int_{\mathbb{R}^N} f(x/2) E_k(-\frac{i}{\sqrt{2}} x, \sqrt{2}z) e^{-|x|^2/4} e^{|z|^2} w_k(x)\, dx \\ &= \tilde{c}\tilde{\tilde{c}} c_k 2^{\gamma + N/2} e^{-|z|^2/2} \int_{\mathbb{R}^N} f(y) E_k(\sqrt{2}y, \sqrt{2}(-iz)) e^{-(|y|^2 + |-iz|^2)} w_k(y)\, dy \\ &= \tilde{c} c_k 2^{\gamma + N/2} \mathscr{B}_k f(-iz) = \mathscr{B}_k f(-iz). \end{aligned}$$

□

Remark 4.8. The above theorem gives a simple alternative proof for the unitarity of the transform $\mathscr{D}_k$, which was proved earlier by Dunkl [12] using a different approach. See also [21].

For $f \in \mathscr{L}^2(\mathbb{R}^N, w_k)$, define the Fourier transform of f by

$$\mathbb{F}_k(f)(\xi) = \int_{\mathbb{R}^N} f(x) E_k(i\xi, x) w_k(x)\, dx.$$

The integral transform $\mathbb{F}_k$ is also know as the inverse of the Dunkl transform, up to a constant. The following theorem is similar to the main result in [34, Section 9], where the statement was proved[1] only in a special setting associated with the flat symmetric spaces of type C_N and D_N ($N \geq 3$).

[1]In [34, Proposition 9.4] the centered formula should not contains the term $(-1)^{-|\underline{\mathbf{n}}|}$, and therefore the expansion in [34, Corollary 9.5 bis] does not contain $(-1)^{-|\underline{\mathbf{n}}|}$.

Theorem 4.9. *For* $x, y \in \mathbb{R}^N$

$$e^{-|x|^2/4}e^{|y|^2/2}E_k(x, iy) = c_k^{-1/2} \sum_{\mathbf{m}\in\mathbb{Z}_+^N} \varphi_{\mathbf{m}}(y)\mathbb{F}_k(\mathscr{B}_k^{-1}\varphi_{\mathbf{m}})(x).$$

Proof. On one hand, by the generating relation (4.1), we know that

$$e^{-|z|^2/2}E_k(x, z) = \sum_{\mathbf{m}\in\mathbb{Z}_+^N} \varphi_{\mathbf{m}}(z)H_{\mathbf{m}}(x).$$

Hence

$$\begin{aligned} e^{-|x|^2/4}e^{|y|^2/2}E_k(x, iy) &= \sum_{\mathbf{m}\in\mathbb{Z}_+^N} i^{|\mathbf{m}|}\varphi_{\mathbf{m}}(y)h_{\mathbf{m}}(x) \\ &= 2^{-\gamma-N/2}c_k^{1/2} \sum_{\mathbf{m}\in\mathbb{Z}_+^N} i^{|\mathbf{m}|}\varphi_{\mathbf{m}}(y)\Psi_{\mathbf{m}}(\frac{x}{2}). \end{aligned} \tag{4.2}$$

On the other hand, we claim that

$$\Psi_{\mathbf{m}}(\frac{x}{2}) = i^{-|\mathbf{m}|}2^{\gamma+N/2}c_k^{-1}\mathbb{F}_k(\Psi_{\mathbf{m}})(x).$$

Therefore, we may rewrite (4.2) as

$$\begin{aligned} e^{-|x|^2/4}e^{|y|^2/2}E_k(x, iy) &= c_k^{-1/2} \sum_{\mathbf{m}\in\mathbb{Z}_+^N} \varphi_{\mathbf{m}}(y)\mathbb{F}_k(\Psi_{\mathbf{m}})(x) \\ &= c_k^{-1/2} \sum_{\mathbf{m}\in\mathbb{Z}_+^N} \varphi_{\mathbf{m}}(y)\mathbb{F}_k(\mathscr{B}_k^{-1}\varphi_{\mathbf{m}})(x). \end{aligned}$$

To prove the claim, recall that $\mathscr{B}_k \circ \mathscr{D}_k \circ \mathscr{B}_k^{-1} = (-i)^*$ (see Theorem 4.7). Therefore

$$\begin{aligned} \mathscr{D}_k(\Psi_{\mathbf{m}})(-\frac{x}{2}) &= \mathscr{D}_k(\mathscr{B}_k^{-1}\varphi_{\mathbf{m}})(-\frac{x}{2}) \\ &= i^{|\mathbf{m}|}\mathscr{B}_k^{-1}(\varphi_{\mathbf{m}})(\frac{x}{2}) = i^{|\mathbf{m}|}\Psi_{\mathbf{m}}(\frac{x}{2}). \end{aligned}$$

Further, one can check that $\mathscr{D}_k(f)(-\frac{x}{2}) = 2^{\gamma+N/2}c_k^{-1}\mathbb{F}_k(f)(x)$. Thus, the claim holds. ◻

Recall that M_ξ denotes the operator $M_\xi(f)(z) = \langle z, \xi\rangle f(z)$, for $\xi \in \mathbb{C}^N$. Define the lowering and the raising operators on $\mathscr{L}^2(\mathbb{R}^N, w_k)$ by

$$A_\xi^- = \frac{1}{\sqrt{2}}(M_{2\xi} + T_\xi(k)), \qquad A_\xi^+ = \frac{1}{\sqrt{2}}(M_{2\xi} - T_\xi(k)).$$

These two operators were introduced by Rösler [31] in connection with Rodrigues-type formulas for the eigenfunctions of the Calogero-Moser systems. Next we will see that these two operators are also the lowering and raising operators on $\mathscr{F}_k(\mathbb{C}^N)$ but in a more natural way.

Below, we will exhibit the relationship between operators on $\mathscr{L}^2(\mathbb{R}^N, w_k)$ and on $\mathscr{F}_k(\mathbb{C}^N)$. For an operator $\mathbb{O}$ on $\mathscr{L}^2(\mathbb{R}^N, w_k)$, we define the operator $\check{\mathbb{O}}$ on $\mathscr{F}_k(\mathbb{C}^N)$ by

$$\check{\mathbb{O}} = \mathscr{B}_k \circ \mathbb{O} \circ \mathscr{B}_k^{-1}.$$

Further, as usual, $[A,B]=AB-BA$ for $A,B\in\mathrm{End}(\mathscr{P}(\mathbb{C}^N))$.

Theorem 4.10. *The following properties hold:*

(i) $\check{T}_\xi(k)=T_\xi(k)-M_\xi$ *for* $\xi\in\mathbb{C}^N$;

(ii) $[\check{T}_\xi(k),\check{T}_\eta(k)]=0$ *for* $\xi,\eta\in\mathbb{C}^N$;

(iii) $\check{M}_{2\xi}=T_\xi(k)+M_\xi$ *for* $\xi\in\mathbb{C}^N$;

(iv) $[\check{M}_{2\xi},\check{M}_{2\eta}]=0$ *for* $\xi,\eta\in\mathbb{C}^N$;

(v) $[\check{T}_\xi(k),\check{M}_{2\eta}]=2\langle\xi,\eta\rangle+2\sum_{\alpha\in R^+}k_\alpha\langle\alpha,\xi\rangle\langle\alpha,\eta\rangle r_\alpha$;

(vi) $\check{A}_\xi^-=\sqrt{2}T_\xi(k)$, *and* $\check{A}_\xi^+=\sqrt{2}M_\xi$.

Notice that, as the Dunkl operators are homogeneous of degree -1 *on polynomials, and since* M_ξ *are the multiplication operators,* $\check{A}_\xi^-$ *and* $\check{A}_\xi^+$ *are obviously the lowering and raising operators on* $\mathscr{P}(\mathbb{C}^N)$.

Proof. (i) Write $\tilde{\tilde{c}}=c_k^{-1/2}2^{\gamma+N/2}$. Using (2.1), we obtain

$$\begin{aligned}
T_\xi^z(k)\mathscr{B}_kf(z) &= T_\xi^z(k)\left[\tilde{\tilde{c}}e^{-|z|^2/2}\int_{\mathbb{R}^N}f(x)e^{-|x|^2/2}E_k(2x,z)w_k(x)dx\right]\\
&= -\tilde{\tilde{c}}\langle z,\xi\rangle e^{-|z|^2/2}\int_{\mathbb{R}^N}f(x)e^{-|x|^2/2}E_k(2x,z)w_k(x)dx\\
&\quad +2\tilde{\tilde{c}}e^{-|z|^2/2}\int_{\mathbb{R}^N}f(x)\langle x,\xi\rangle e^{-|x|^2/2}E_k(2x,z)w_k(x)dx\\
&= -\langle z,\xi\rangle\mathscr{B}_k(f)(z)+2\tilde{\tilde{c}}e^{-|z|^2/2}\int_{\mathbb{R}^N}f(x)\langle x,\xi\rangle e^{-|x|^2/2}E_k(2x,z)w_k(x)dx.
\end{aligned}$$

Further, using again (2.1), we prove that

$$\mathscr{B}_k(T_\xi(k)f)(z) = -2\langle z,\xi\rangle\mathscr{B}_k(f)(z)+2\tilde{\tilde{c}}e^{-|z|^2/2}\int_{\mathbb{R}^N}f(x)\langle x,\xi\rangle e^{-|x|^2/2}E_k(2x,z)w_k(x)dx.$$

Hence

$$T_\xi^z(k)\mathscr{B}_k(f)(z)=\mathscr{B}_k(T_\xi(k)f)(z)+\langle z,\xi\rangle\mathscr{B}_k(f)(z),$$

and statement (i) holds.

(ii) Since the Dunkl operators commute, one can derive directly the statement from the definition of $\check{T}_\xi(k)=\mathscr{B}_k\circ T_\xi(k)\circ\mathscr{B}_k^{-1}$.

(iii) From the proof of (i), it follows that

$$T_\xi^z(k)\mathscr{B}_kf(z)=-\langle z,\xi\rangle\mathscr{B}_k(f)(z)+2\tilde{\tilde{c}}e^{-|z|^2/2}\int_{\mathbb{R}^N}f(x)\langle x,\xi\rangle e^{-|x|^2/2}E_k(2x,z)w_k(x)dx,$$

and therefore

$$\mathscr{B}_k(2\langle\xi,\cdot\rangle f)(z)=T_\xi^z(k)\mathscr{B}_kf(z)+\langle z,\xi\rangle\mathscr{B}_k(f)(z).$$

(iv) The statement follows by employing the same argument used in (ii)

(v) Using again the commutativity of $T_\xi(k)$, we obtain

$$[\check{T}_\xi(k),\check{M}_{2\eta}]=T_\xi(k)\circ\langle\eta,\cdot\rangle-\langle\eta,\cdot\rangle T_\xi(k)+T_\eta(k)\circ\langle\xi,\cdot\rangle-\langle\xi,\cdot\rangle T_\eta(k).$$

Now the statement follows from the following (see (3.2))

$$T_\xi(k)\{\langle\eta, z\rangle f(z)\} = \langle\eta, \xi\rangle f(z) + \langle\eta, z\rangle T_\xi(k) f(z) + \sum_{\alpha\in R^+} k_\alpha\langle\alpha, \xi\rangle\langle\alpha, \eta\rangle f(r_\alpha z),$$

by symmetry in ξ and η.

(vi) Follows from the assertions (i) and (iii). □

The quantum Calogero-Moser (CM) rational model describes quantum mechanical systems of N particles in one dimension identified by their coordinates and interacting pairwise through potentials of type $1/A^2$.

The generalized CM operator related to a Coxeter group G and a multiplicity function k was introduced by Olshanetsky and Perelomov [24]. The gauge equivalent version of the Hamiltonian of such a system with harmonic confinement has the form

$$\tilde{\mathscr{H}}_k := w_k^{-1/2}(-\tilde{\mathscr{L}}_k + \omega^2|x|^2)w_k^{1/2} = -\Delta + 2\sum_{\alpha\in R^+}\frac{k_\alpha}{\langle\alpha, x\rangle}\partial_\alpha + \omega^2|x|^2, \quad \omega > 0,$$

where

$$\tilde{\mathscr{L}}_k = \Delta - 2\sum_{\alpha\in R^+}\frac{1}{\langle\alpha, x\rangle^2}k_\alpha(k_\alpha - 1).$$

If R is of type A_{N-1}, the study of $\tilde{\mathscr{H}}_k$ goes back to Calogero [4]. In the original formulation of the CM model, the interaction among the particles was simply pairwise. It was realized later that the complete integrability of the model was tied to the root lattice of the Lie algebra of type A_{N-1}. For an arbitrary root system on $\mathbb{R}^N$, partial results on the integrability of the generalized CM system are due to Olshanetsky and Perelomov [24]. See also [25]. A new approach in the understanding of the algebraic structure and the quantum integrability of CM models was later discovered by Heckman using the Dunkl operators [18]. We recall briefly this approach. Let

$$\mathscr{L}_k = \Delta - 2\sum_{\alpha\in R^+}\frac{1}{\langle\alpha, x\rangle^2}k_\alpha(k_\alpha - r_\alpha).$$

Its gauge equivalent version is given by $w_k^{-1/2}\mathscr{L}_k w_k^{1/2} = \Delta_k$ (cf. [29]). Let $\mathscr{P}(\mathbb{R}^N)^G$ be the set of G-invariant polynomials on $\mathbb{R}^N$. For $p \in \mathscr{P}(\mathbb{R}^N)^G$ we denote by $\mathrm{Res}(p(T))$ the restriction of the Dunkl operator $p(T)$ to $\mathscr{P}_k(\mathbb{R}^N)^G$. By [18, Theorem 1.7], the set $S = \{\mathrm{Res}(p(T)) \mid p \in \mathscr{P}(\mathbb{R}^N)^G\}$ is a commuting family of differential operators on $\mathscr{P}(\mathbb{R}^N)^G$ containing the operator $\mathrm{Res}(\Delta_k) = w_k^{-1/2}\tilde{\mathscr{L}}_k w_k^{1/2}$, and S has N algebraically independent generators. This implies the integrability of the CM operators $\tilde{\mathscr{H}}_k$.

Consider the following gauge equivalent version

$$\mathscr{H}_k := \frac{1}{4}w_k^{-1/2}(-\mathscr{L}_k + 4|x|^2)w_k^{1/2} = \frac{1}{4}(-\Delta_k + 4|x|^2)$$

of the CM Hamiltonian with harmonic confinement and reflection terms. We choose the constants 1/4 and 4 in the definition above to simplify formulas. The operator $\mathscr{H}_k$ is densely defined in $\mathscr{L}^2(\mathbb{R}^n, w_k)$. For instance, $d_2 \circ h_{\mathbf{m}}$ (here d_2 denotes the delation operator) is an eigenfunction for $\mathscr{H}_k$, for the eigenvalue $|\mathbf{m}| + \gamma + N/2$ (cf. [30]).

Theorem 4.11. *Let $\{\xi_1, \ldots, \xi_N\}$ be any orthonormal basis of $\mathbb{C}^N$. On $\mathscr{F}_k(\mathbb{C}^N)$, the corresponding operator to the Hamiltonian $\mathscr{H}_k$ is given by*

$$\check{\mathscr{H}}_k = (\gamma + N/2) + \sum_{i=1}^{N} \xi_i \partial_{\xi_i}.$$

Proof. By Theorem 4.10 we have

$$\check{T}_\xi(k)^2 = T_\xi(k)^2 + \langle \xi, \cdot \rangle^2 - \langle \xi, \cdot \rangle T_\xi(k) - T_\xi(k) \circ \langle \xi, \cdot \rangle,$$

and

$$\check{M}_{2\xi}^2 = T_\xi(k)^2 + \langle \xi, \cdot \rangle^2 + \langle \xi, \cdot \rangle T_\xi(k) + T_\xi(k) \circ \langle \xi, \cdot \rangle.$$

Therefore

$$-\check{T}_\xi(k)^2 + \check{M}_{2\xi}^2 \;=\; 2\langle \xi, \cdot \rangle T_\xi(k) + 2T_\xi(k) \circ \langle \xi, \cdot \rangle.$$

On the other hand, for $\xi \in \mathbb{C}^N$ such that $|\xi| = 1$, we have

$$\begin{aligned}
& T_\xi(k)(\langle \xi, z \rangle f(z)) \\
= \;& \partial_\xi(\langle \xi, z \rangle f(z)) + \sum_{\alpha \in R^+} k_\alpha \frac{\langle \alpha, \xi \rangle}{\langle \alpha, z \rangle} (\langle \xi, z \rangle f(z) - \langle \xi, r_\alpha z \rangle f(r_\alpha z)) \\
= \;& f(z) + \langle \xi, z \rangle \partial_\xi f(z) + \sum_{\alpha \in R^+} k_\alpha \frac{\langle \alpha, \xi \rangle \langle z, \xi \rangle}{\langle \alpha, z \rangle} f(z) - \sum_{\alpha \in R^+} k_\alpha \frac{\langle \alpha, \xi \rangle \langle r_\alpha z, \xi \rangle}{\langle \alpha, z \rangle} f(r_\alpha z).
\end{aligned}$$

Hence

$$\begin{aligned}
-\check{T}_\xi(k)^2 f(z) + \check{M}_{2\xi}^2 f(z) \;=\; & 2f(z) + 4\langle z, \xi \rangle \partial_\xi f(z) + 4 \sum_{\alpha \in R^+} k_\alpha \frac{\langle \alpha, \xi \rangle \langle z, \xi \rangle}{\langle \alpha, z \rangle} f(z) \\
& -2 \sum_{\alpha \in R^+} k_\alpha \frac{\langle \alpha, \xi \rangle \langle r_\alpha z, \xi \rangle}{\langle \alpha, z \rangle} f(r_\alpha z) - 2 \sum_{\alpha \in R^+} k_\alpha \frac{\langle \alpha, \xi \rangle \langle z, \xi \rangle}{\langle \alpha, z \rangle} f(r_\alpha z).
\end{aligned}$$

Using the following Parseval identity

$$\sum_{i=1}^{N} \langle \xi_i, z \rangle \langle \xi_i, w \rangle = \langle z, w \rangle,$$

and the fact that $\langle r_\alpha z, \alpha \rangle = -\langle z, \alpha \rangle$, we obtain

$$\sum_{i=1}^{N} -\check{T}_{\xi_i}(k)^2 f(z) + \check{M}_{2\xi_i}^2 f(z) = 4 \left\{ \sum_{i=1}^{N} \langle z, \xi_i \rangle \partial_{\xi_i} f(z) + (\gamma + N/2) f(z) \right\}.$$

$\square$

Remark 4.12.

(i) An operator $\mathbb{O}$ is called essentially self-adjoint, if it is symmetric and its closure is self-adjoint. Let $\mathbb{O}$ be a symmetric operator on a Hilbert space $\mathscr{H}$ with domain $\mathbb{D}(\mathbb{O})$, and let $\{f_n\}_n$ be a complete orthonormal set in $\mathscr{H}$. If each $f_n \in \mathbb{D}(\mathbb{O})$ and there exists $\lambda_n \in \mathbb{R}$ such that $\mathbb{O}f_n = \lambda_n f_n$, for every n, then $\mathbb{O}$ is essentially self-adjoint and the spectrum of its closure $\overline{\mathbb{O}}$, which is a self-adjoint operator, is given by $\mathrm{Spec}(\overline{\mathbb{O}}) = \{\lambda_n \mid n \in \mathbb{Z}^+\}$. We refer to [6, Chapter 1] for more details on this matter. Now, using Theorem 3.7, one can see that $\check{\mathscr{H}}_k$ is densely defined and symmetric in $\mathscr{F}_k(\mathbb{C}^N)$ while the set $\{\varphi_{\mathbf{m}} \,|\, \mathbf{m} \in \mathbb{Z}_+^N\}$ forms an orthonormal basis for $\mathscr{F}_k(\mathbb{C}^N)$ with $\check{\mathscr{H}}_k\varphi_{\mathbf{m}} = (\gamma + N/2 + |\mathbf{m}|)\varphi_{\mathbf{m}}$. Therefore, from the above discussion, the operator $\check{\mathscr{H}}_k$ is essentially self-adjoint and

$$\mathrm{Spec}(\overline{\check{\mathscr{H}}}_k) = \left\{ \ell + \gamma + N/2 \mid \ell \in \mathbb{Z}^+ \right\}.$$

Clearly, the study of the Hamiltonian $\mathscr{H}_k$ in the Fock model is rather easy.

(ii) Since $\check{\mathscr{H}}_k\varphi_{\mathbf{m}}(x) = (\gamma + N/2 + |\mathbf{m}|)\varphi_{\mathbf{m}}(x)$, then

$$(-i)^*\varphi_{\mathbf{m}}(x) = (-i)^{|\mathbf{m}|}\varphi_{\mathbf{m}}(x) = e^{-i\frac{\pi}{2}(\check{\mathscr{H}}_k - (\gamma+N/2))}\varphi_{\mathbf{m}}(x).$$

Using the fact that $(-i)^* = \mathscr{B}_k \circ \mathscr{D}_k \circ \mathscr{B}_k^{-1}$ and $\check{\mathscr{H}}_k = \mathscr{B}_k \circ \left[\frac{1}{4}(-\Delta_k + 4|x|^2)\right] \circ \mathscr{B}_k^{-1}$, one may write the Dunkl transform $\mathscr{D}_k$ as

$$\mathscr{D}_k = e^{i\frac{\pi}{2}(\gamma+N/2)}e^{-i\frac{\pi}{8}(-\Delta_k+4|x|^2)}.$$

Remark 4.13.

The generalized Fock space theory presented in this paper occurs also for product Fock spaces. Fix a Coxeter group G on $\mathbb{R}^N$ with root system R. Let ℓ be a positive integer and let $\underline{k} = (k_1, \dots, k_\ell)$ be a collection of ℓ non-negative multiplicity functions. For $z^{(1)}, \dots, z^{(\ell)} \in \mathbb{C}^N$, let $\underline{z} = (z^{(1)}, \dots, z^{(\ell)}) \in \mathbb{C}^{N\times\ell}$. For $\underline{z}, \underline{w} \in \mathbb{C}^{N\times\ell}$, we define

$$\mathbb{K}_{\underline{k}}(\underline{z}, \underline{w}) := \mathbb{K}_{k_1}(z^{(1)}, w^{(1)}) \cdots \mathbb{K}_{k_\ell}(z^{(\ell)}, w^{(\ell)}),$$

where $\mathbb{K}_{k_i}(z^{(i)}, w^{(i)}) = E_{k_i}(z^{(i)}, \overline{w^{(i)}})$. Let $\mathbb{S}_{\underline{k}}(\mathbb{C}^{N\times\ell})$ be the set of all finite complex linear combinations

$$f = \sum_{i=1}^{n} \alpha_i \mathbb{K}_{\underline{k}}(\cdot, \underline{z}^{(i)}), \qquad \alpha_i \in \mathbb{C}, \quad \underline{z}^{(i)} \in \mathbb{C}^{N\times\ell}.$$

The completion of $\mathbb{S}_{\underline{k}}(\mathbb{C}^{N\times\ell})$ with respect to the norm

$$\|f\|_{\underline{k}}^2 = \langle\langle f, f\rangle\rangle_{\underline{k}} := \sum_{i=1}^{n} |\alpha_i|^2 \mathbb{K}_{\underline{k}}(\underline{z}^{(i)}, \underline{z}^{(i)})$$

coincides with the Hilbert space $\mathscr{F}_{\underline{k}}(\mathbb{C}^{N\times\ell})$ of holomorphic functions on $\mathbb{C}^{N\times\ell}$ with reproducing kernel $\mathbb{K}_{\underline{k}}(\underline{z}, \underline{w})$. Here $\mathscr{F}_{\underline{k}}(\mathbb{C}^{N\times\ell})$ will be the Fock space related to the root system R^ℓ and to the reflection group G^ℓ. Moreover, on $\mathscr{P}(\mathbb{C}^{N\times\ell})$, the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\underline{k}}$ is given by

$$\langle\!\langle p,q\rangle\!\rangle_{\underline{k}} = p(T(k_1),\ldots,T(k_\ell))\bar{q}(\underline{z})\Big|_{\underline{z}=0}, \qquad p,q\in\mathscr{P}(\mathbb{C}^{N\times\ell}),$$

where $T(k_i) = (T_1(k_i),\ldots,T_N(k_i))$, and $p(T(k_1),\ldots,T(k_\ell))$ is the operator derived from $p(\underline{z})$ by replacing $z_{i,j}$ by $T_i(k_j)$.

For $\underline{x} = (x^{(1)},\ldots,x^{(\ell)}) \in \mathbb{R}^{N\times\ell}$, put $w_{\underline{k}}(\underline{x}) = w_{k_1}(x^{(1)})\cdots w_{k_\ell}(x^{(\ell)})$, and let $\mathscr{L}^2(\mathbb{R}^{N\times\ell}, w_{\underline{k}})$ be the space of $\mathscr{L}^2$-functions on $\mathbb{R}^{N\times\ell}$ with respect to the measure $w_{\underline{k}}(\underline{x})d\underline{x}$. In particular, if $\underline{\mathbf{m}} = (\mathbf{m}^{(1)},\ldots,\mathbf{m}^{(\ell)}) \in \mathbb{Z}_+^{N\times\ell}$ with $\mathbf{m}^{(i)}\in\mathbb{Z}_+^N$, then the set $\{\Psi_{\underline{\mathbf{m}}} \mid \underline{\mathbf{m}}\in\mathbb{Z}_+^{N\times\ell}\}$, where

$$\Psi_{\underline{\mathbf{m}}}(\underline{x}) = \prod_{i=1}^{\ell} 2^{\gamma_i+N/2} c_{k_i}^{-1/2} h_{\mathbf{m}^{(i)}}(2x^{(i)})$$

forms an orthonormal basis for $\mathscr{L}^2(\mathbb{R}^{N\times\ell}, w_{\underline{k}})$. Finally, for $f\in\mathscr{L}^2(\mathbb{R}^{N\times\ell}, w_{\underline{k}})$ and $\underline{z}\in\mathbb{C}^{N\times\ell}$, let

$$\mathscr{B}_{\underline{k}}f(\underline{z}) = c(k,\ell)e^{-\mathrm{tr}(\underline{z}\underline{z}^t)}\int_{\mathbb{R}^{N\times\ell}} f(\underline{x})E_{\underline{k}}(\sqrt{2}\underline{x},\sqrt{2}\underline{z})e^{-\mathrm{tr}(\underline{x}\underline{x}^t)}w_{\underline{k}}(\underline{x})d\underline{x},$$

where $c(k,\ell) = 2^{\sum_{i=1}^{\ell}\gamma_i+N\ell/2}\Big(\prod_{i=1}^{\ell} c_{k_i}^{-1/2}\Big)$, A^t stands for the transpose of a matrix A, and $\mathrm{tr}(A)$ denotes its trace. The integral transform $\mathscr{B}_{\underline{k}}$ is the Segal-Bargmann transform associated with the Coxeter group G^ℓ.

With these definitions and basic results, one can derive the results detailed in this paper for the Fock space $\mathscr{F}_{\underline{k}}(\mathbb{C}^{N\times\ell})$.

Example 4.14. Assume that R is a rank one root system of type A_1, i.e. $R = \{\pm\sqrt{2}\alpha\}$. The Coxeter group G reduces to $\{\pm1\}\simeq\mathbb{Z}/2\mathbb{Z}$, and acts on $\mathbb{C}$ by multiplication. For all complex numbers c we adopt the identification $\alpha(c)\equiv c$. The Dunkl operator associated with a multiplicity parameter $k\in\mathbb{C}$ is given by

$$T(k)f(x) = f'(x) + \frac{k}{x}(f(x)-f(-x)), \qquad x\in\mathbb{R}.$$

The Dunkl-kernel E_k is given by

$$E_k(z,w) = \Gamma(k+\frac{1}{2})\left(\frac{zw}{2}\right)^{\frac{1}{2}-k}\left\{I_{k-\frac{1}{2}}(zw) + I_{k+\frac{1}{2}}(zw)\right\},$$

where $I_\nu(z) = e^{-i\pi\nu/2}J_\nu(iz)$, $J_\nu(z)$ being the Bessel function of the first kind

$$J_\nu(z) = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell\left(\frac{z}{2}\right)^{2\ell+\nu}}{\Gamma(1+\nu+\ell)\ell!}.$$

In this example

$$c_k = 2^{k-\frac{1}{2}}\Gamma(k+\frac{1}{2}), \quad \text{and} \quad w_k(x) = |x|^{2k}, \qquad k\geq 0.$$

We refer to [28] for a thorough study of the space $\mathscr{L}^2(\mathbb{R}, |x|^{2k})$.

In this example, we will see that the norm $\|\cdot\|_k$ can be written explicitly as an $\mathscr{L}^2$-norm. Our idea uses a result by F.M. Cholewinski [5], where the author considers only the Fock space $\mathscr{F}_k^e(\mathbb{C})$ of even entire functions on $\mathbb{C}$. In [5], the author has also investigated the Segal-Bargmann transform on $\mathscr{F}_k^e(\mathbb{C})$ from another point of view.

Let $\mathscr{F}_k^e(\mathbb{C})$ be the set of even entire functions with inner product

$$\langle\langle f, g\rangle\rangle_{k,e} = \int_{\mathbb{C}} f(z)\overline{g(z)}\,d\mu_k^e(z),$$

where

$$d\mu_k^e(z) = \frac{\|z\|^{2k+1}}{2^{k-1/2}\pi\Gamma(k+1/2)}\mathscr{K}_{k-1/2}(\|z\|^2)dz, \qquad (\|z\|^2 = z\bar{z})$$

with

$$\mathscr{K}_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu}$$

is the Bessel function of the third kind. In [5], the author proves that

$$\mathbb{K}^e(z, w) := \Gamma(k + \frac{1}{2})\left(\frac{z\bar{w}}{2}\right)^{\frac{1}{2}-k} I_{k-\frac{1}{2}}(z\bar{w})$$

is the reproducing kernel of $\mathscr{F}_k^e(\mathbb{C})$.

Now, using a similar idea, we consider $\mathscr{F}_k^o(\mathbb{C})$ to be the set of odd entire functions with inner product

$$\langle\langle f, g\rangle\rangle_{k,o} = \int_{\mathbb{C}} f(z)\overline{g(z)}\,d\mu_k^o(z),$$

where

$$d\mu_k^o(z) = \frac{\|z\|^{2k+1}}{2^{k-1/2}\pi\Gamma(k+1/2)}\mathscr{K}_{k+1/2}(\|z\|^2)dz.$$

Therefore, we can show that

$$\mathbb{K}^o(z, w) := \Gamma(k + \frac{1}{2})\left(\frac{z\bar{w}}{2}\right)^{\frac{1}{2}-k} I_{k+\frac{1}{2}}(z\bar{w})$$

is the reproducing kernel of $\mathscr{F}_k^o(\mathbb{C})$.

Set $EL^2(\mathbb{C}) := \mathscr{F}_k^e(\mathbb{C}) \oplus \mathscr{F}_k^o(\mathbb{C})$. Using an elementary argument on reproducing kernels, we can conclude that the reproducing kernel of $EL^2(\mathbb{C})$ is given by

$$\mathbb{K}_k(z, w) = \Gamma(k + \frac{1}{2})\left(\frac{z\bar{w}}{2}\right)^{\frac{1}{2}-k}\left\{I_{k-\frac{1}{2}}(z\bar{w}) + I_{k+\frac{1}{2}}(z\bar{w})\right\},$$

which is equal to $E_k(z, \bar{w})$. Since the Hilbert space $EL^2(\mathbb{C})$ is uniquely determined by its reproducing kernel $\mathbb{K}_k(z, w)$, which coincides with $E_k(z, \bar{w})$, it follows that $EL^2(\mathbb{C})$ is the Fock space $\mathscr{F}_k(\mathbb{C})$ introduced in this paper.

In conclusion, for $N = 1$, the measure associated with $\mathscr{F}_k(\mathbb{C})$ is given by

$$d\mu_k(z) = \frac{\|z\|^{2k+1}}{2^{k-1/2}\pi\Gamma(k+1/2)}\left\{\mathscr{K}_{k-1/2}(\|z\|^2)\Big|_{\text{even part}} + \mathscr{K}_{k+1/2}(\|z\|^2)\Big|_{\text{odd part}}\right\}dz, \tag{4.3}$$

in the sense that $f(z) = \left[\frac{f(z)+f(-z)}{2}\right] + \left[\frac{f(z)-f(-z)}{2}\right]$. The set $\{\varphi_m \mid m \in \mathbb{Z}^+\}$, where

$$\varphi_m(z) = \frac{z^m}{\gamma_k(m)^{1/2}}$$

with

$$\gamma_k(2m) = \frac{2^{2m} m!\Gamma(m+k+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}, \qquad \gamma_k(2m+1) = \frac{2^{2m+1} m!\Gamma(m+k+\frac{3}{2})}{\Gamma(k+\frac{1}{2})},$$

forms an orthonormal basis for $\mathscr{F}_k(\mathbb{C})$. The Segal-Bargmann transform $\mathscr{B}_k$ is given by

(4.4)
$$\mathscr{B}_k f(z) = 2^{k/2+3/4}\Gamma(k+\frac{1}{2})^{1/2} z^{1/2-k} e^{-z^2/2} \int_{\mathbb{R}} f(x)\big(I_{k-\frac{1}{2}}(2zx) + I_{k+\frac{1}{2}}(2zx)\big) e^{-x^2} |x|^{k+1/2} dx.$$

Example 4.15. Denote by $\{e_1, \ldots, e_N\}$ the standard basis of $\mathbb{R}^N$. Let $G = (\mathbb{Z}/2\mathbb{Z})^N$ be the Coxeter group generated by the reflection $r_1, \ldots, r_N$ along $e_1, \ldots, e_N$, i.e. for $x \in \mathbb{R}^N$ $r_i(x) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_N)$.

Let $\underline{k} = (k_1, \ldots, k_N)$ with $k_i \geq 0$. By Remark 4.13 and Example 4.14, the Fock space $\mathscr{F}_{\underline{k}}(\mathbb{C}^N)$ related to the reflection group $(\mathbb{Z}/2\mathbb{Z})^N$, is the Hilbert space of holomorphic functions on $\mathbb{C}^N$ with inner product

$$\langle\langle f, g\rangle\rangle_{\underline{k}} = \int_{\mathbb{C}^N} f(\underline{z})\overline{g(\underline{z})} d\mu_{\underline{k}}(\underline{z}),$$

where $\underline{z} = (z_1, \ldots, z_N) \in \mathbb{C}^N$, and $d\mu_{\underline{k}}(\underline{z}) = d\mu_{k_1}(z_1)\cdots d\mu_{k_N}(z_N)$ with $d\mu_{k_i}(z_i)$ is the measure given by (4.3). Moreover, for $\underline{z}, \underline{w} \in \mathbb{C}^N$, the reproducing kernel $\mathbb{K}_{\underline{k}}(\underline{z}, \underline{w})$ is given by

$$\mathbb{K}_{\underline{k}}(\underline{z}, \underline{w}) = \prod_{i=1}^N \Gamma(k_i + \frac{1}{2}) \left(\frac{z_i \bar{w}_i}{2}\right)^{\frac{1}{2}-k_i} \left\{ I_{k_i-\frac{1}{2}}(z_i\bar{w}_i) + I_{k_i+\frac{1}{2}}(z_i\bar{w}_i)\right\}.$$

The unitary isomorphism $\mathscr{B}_{\underline{k}} : \mathscr{L}^2(\mathbb{R}^N, \prod_{i=1}^N |x_i|^{2k_i}) \to \mathscr{F}_{\underline{k}}(\mathbb{C}^N)$ is given by $\mathscr{B}_{\underline{k}}(f)(\underline{z}) = \mathscr{B}_{k_1} \otimes \mathscr{B}_{k_2} \otimes \cdots \otimes \mathscr{B}_{k_N}(f)(\underline{z})$, where the transforms $\mathscr{B}_{k_i}$ are given by (4.4). Further, the set $\{\varphi_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{Z}_+^N\}$ with

$$\varphi_{\mathbf{m}}(\underline{z}) = \prod_{i=1}^N \frac{z_i^{m_i}}{\gamma_{k_i}(m_i)^{1/2}}, \qquad m_i \in \mathbb{Z}_+,$$

forms an orthonormal basis for $\mathscr{F}_{\underline{k}}(\mathbb{C}^N)$.

Remark 4.16.

(i) Clearly $\mathbf{K}(\underline{k}, \underline{z}, \underline{w}) = \prod_{i=1}^N (z_i w_i)^{1/2-k_i} \mathscr{K}_{k_i-1/2}(z_i w_i)$ is a solution of the Bessel differential system (3.7) for $G = (\mathbb{Z}/2\mathbb{Z})^N$, with the required asymptotic behavior at infinity.

(ii) Define ℓ_2 to be the set of sequences $\{z_n\}_{n=1}^\infty$ of complex numbers which satisfy

$\sum_{n=1}^{\infty} \|z_n\|^2 < \infty$. In Example 4.15, one may let $N \to \infty$. The resulting Fock space is the Hilbert space $\mathscr{F}_\infty(\ell_2)$ of "holomorphic" functions on ℓ_2 such that

$$\|f\|_\infty^2 := \int_{\mathbb{C}^\infty} |f(z)|^2 d\mu(z) < \infty.$$

The measure $d\mu$ on $\mathbb{C}^\infty$ is defined as the product of the measures $d\mu_{k_i}$, given by (4.3), on each component. For $z, w \in \ell_2$, the reproducing kernel of $\mathscr{F}_\infty(\ell_2)$ is given by

$$E_\infty(z,w) := \prod_{i=1}^{\infty} \Gamma(k_i + \frac{1}{2}) \left(\frac{z_i \bar{w}_i}{2}\right)^{\frac{1}{2}-k_i} \left\{ I_{k_i - \frac{1}{2}}(z_i \bar{w}_i) + I_{k_i+\frac{1}{2}}(z_i \bar{w}_i) \right\}.$$

Similarly, other facts proved for $\mathscr{F}_{\underline{k}}(\mathbb{C}^N)$, with $G = (\mathbb{Z}/2\mathbb{Z})^N$, may be "translated" into corresponding assertions for $\mathscr{F}_\infty(\ell_2)$.

5. A BRANCHING DECOMPOSITION OF THE FOCK SPACE AND HECKE'S TYPE FORMULA

This section describes the structure of a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ (or the universal covering $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$) on $\mathscr{P}(\mathbb{C}^N)$. This Lie algebra representation, together with the left regular action of the Coxeter group G, allows to obtain the branching decomposition of the Fock space under the action of $G \times \mathfrak{sl}(2,\mathbb{R})$. Those readers who are familiar with the theory of Howe reductive dual pairs [19, 20] will find that our formulation can be thought of as an instance of this theory. The Hecke's formula for the Dunkl transform holds immediately from our $\mathfrak{sl}(2,\mathbb{R})$-representation.

Choose $z_1, z_2, \dots, z_N$ as a system of coordinates on $\mathbb{C}^N$. Let

$$E = \frac{1}{2}|z|^2 = \frac{1}{2}\sum_{i=1}^{N} z_i^2, \quad F = -\frac{1}{2}\Delta_k, \quad H = \sum_{i=1}^{N} z_i \partial_{z_i} + N/2 + \gamma.$$

Then E (resp. F) acts on $\mathscr{F}_k(\mathbb{C}^N)$ as a creation (resp. annihilation) operator, and H acts on $\mathscr{F}_k(\mathbb{C}^N)$ as a number operator. If $\mathscr{P}(\mathbb{C}^N) = \bigoplus_{m=0}^{\infty} \mathscr{P}_m(\mathbb{C}^N)$ is the natural grading on $\mathscr{P}(\mathbb{C}^N)$, it is clear that E raises $\mathscr{P}_m(\mathbb{C}^N)$ to $\mathscr{P}_{m+2}(\mathbb{C}^N)$, F lowers $\mathscr{P}_m(\mathbb{C}^N)$ to $\mathscr{P}_{m-2}(\mathbb{C}^N)$, and H multiplies (elementwise) $\mathscr{P}_m(\mathbb{C}^N)$ by the number $(N/2 + \gamma + m)$. In [18], Heckman showed the following commutation relations

$$[E,F] = H, \quad [E,H] = -2E, \quad [F,H] = 2F. \tag{5.1}$$

These are the commutation relations of a standard basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Equation (??) gives raise to a unitary representation ω of $\mathfrak{sl}(2,\mathbb{R})$. On $\mathscr{P}(\mathbb{C}^N)$, the representation ω can be described as

$$\omega(\mathfrak{sl}(2,\mathbb{R})) = \mathfrak{sl}_2^{(2,0)} \oplus \mathfrak{sl}_2^{(1,1)} \oplus \mathfrak{sl}_2^{(0,2)}, \tag{5.2}$$

where

$$\mathfrak{sl}_2^{(2,0)} = \mathrm{Span}\{E\}, \quad \mathfrak{sl}_2^{(1,1)} = \mathrm{Span}\{H\}, \quad \mathfrak{sl}_2^{(0,2)} = \mathrm{Span}\{F\}.$$

The decomposition (??) is an instance of the Cartan decomposition

$$\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{p}^+ \oplus \mathfrak{k} \oplus \mathfrak{p}^-$$

where $\mathfrak{sl}_2^{(2,0)} \simeq \omega(\mathfrak{p}^+)$, $\mathfrak{sl}_2^{(1,1)} \simeq \omega(\mathfrak{k})$, and $\mathfrak{sl}_2^{(0,2)} \simeq \omega(\mathfrak{p}^-)$. Notice that $\mathfrak{k} = \mathfrak{u}(1)$, the Lie algebra of the compact group $U(1)$. The integrated form of the Lie algebra representation ω is the metaplectic representation, or the oscillator representation, of the universal covering $\widetilde{SL(2,\mathbb{R})}$ of the group $SL(2,\mathbb{R})$ (or $Sp(2,\mathbb{R})$). Notice that if γ is an integer, we obtain the metaplectic representation of the double covering $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$. By applying the Segal-Bargmann transform, one obtains the Schrödinger representation of $\widetilde{SL(2,\mathbb{R})}$. However, for our purpose, its infinitesimal action (??) is enough.

Since ω is a unitary representation, and the operator H, which is the generator of $\mathfrak{k}$, has a positive spectrum, then the representation contains a unique vector ν_0 such that $\omega(\mathfrak{p}^-)\nu_0 = 0$ and $\omega(\mathfrak{k})\nu_0 = (m + N/2 + \gamma)\nu_0$ for some positive integer m. The vector ν_0 is the so-called lowest weight vector for the representation, and the number $(m + N/2 + \gamma)$ is the lowest weight. The space of representation then has an orthonormal basis consisting of the vectors $\nu_\ell = \omega(\mathfrak{p}^+)^\ell \nu_0$. It is easy to check that each vector ν_m is an eigenvector for $\omega(\mathfrak{k})$ with eigenvalue $(m + 2\ell + N/2 + \gamma)$. Denote by $\mathscr{W}_{m+\gamma+N/2}$ the representation with lowest weight $m + N/2 + \gamma$.

For $m \in \mathbb{N}$, set $\mathscr{H}_m(\subset \mathscr{P}(\mathbb{C}^N))$ to be the space of harmonic homogeneous polynomials of degree m, i.e. all functions $p \in \mathscr{P}_m(\mathbb{C}^N)$ such that $\Delta_k p = 0$. It is clear that $p \in \mathscr{H}_m$ if and only if $\omega(\mathfrak{k})p = (m + N/2 + \gamma)p$ and $\omega(\mathfrak{p}^-)p = 0$.

Now one of the key features in this formalism is the following branching decomposition.

Theorem 5.1. *The space $\mathscr{P}_m(\mathbb{C}^N)$ of homogeneous polynomials of degree m has a unique decomposition of the form*

$$\mathscr{P}_m(\mathbb{C}^N) = \sum_{t=0}^{[m/2]}{}^{\oplus} |z|^{2t}\mathscr{H}_{m-2t},$$

where $\mathscr{H}_{m-2t}$ denotes the space of harmonic homogeneous polynomials of degree $m - 2t$. Moreover, every homogeneous polynomial $\psi \in \mathscr{P}_m(\mathbb{C}^N)$ can be written in a unique way as

$$\psi(z) = \sum_{t=0}^{[m/2]} \frac{\Gamma(N/2 + m - t + \gamma - 1)}{4^t\Gamma(t+1)\Gamma(N/2 + m + \gamma - 1)} |z|^{2t} h_{m-2t},$$

where $h_{m-2t} \in \mathscr{H}_{m-2t}$ and is given explicitly by

$$h_{m-2t} = \sum_{j=0}^{[m/2]-t} \frac{(-1)^j\Gamma(N/2 + m - 2t - j - 1 + \gamma)}{4^j\Gamma(j+1)\Gamma(N/2 + m - 2t + \gamma - 1)} |z|^{2j}\Delta_k^{t+j}\psi.$$

Proof. If $m = 0$ or 1, the statement is obvious. For the rest of the proof we need the following equation, that may be found in [15]

$$\Delta_k(|z|^{2j}p) = 4j(N/2 + j - 1 + m + \gamma)|z|^{2j-2}p + |x|^{2j}\Delta_k p, \qquad p \in \mathscr{P}_m(\mathbb{C}^N). \tag{5.3}$$

However, one can also derive this equation directly by using the commutation $\left[\Delta_k, |z|^2\right] = 4\check{\mathscr{H}}_k$. Next assume that $m \geq 2$. Define

$$Q_0 p = \sum_{j=0}^{[m/2]} c_{j,m}|z|^{2j}\Delta_k^j p,$$

with $c_{0,m} \neq 0$ and

$$c_{j,m} = \frac{(-1)^j}{4^j}\frac{\Gamma(N/2 + m - j - 1 + \gamma)}{\Gamma(j+1)\Gamma(N/2 + m + \gamma - 1)}c_{0,m}.$$

Here $[m/2]$ denotes the integer part of $m/2$. Notice that $Q_0 p \in \mathscr{P}_m(\mathbb{C}^N)$. Next, we will prove that $Q_0 p \in \mathscr{H}_m$. By using (??), we obtain

$$\begin{aligned}
\Delta_k(Q_0 p) &= \sum_{j=0}^{[m/2]} c_{j,m}\Delta_k(|z|^{2j}\Delta_k^j p) \\
&= \sum_{j=0}^{[m/2]} c_{j,m}4j(N/2 + j - 1 + m - 2j + \gamma)|z|^{2j-2}\Delta_k^j p + \sum_{j=0}^{[m/2]} c_{j,m}|z|^{2j}\Delta_k^{j+1}p \\
&= \sum_{j=1}^{[m/2]} c_{j,m}4j(N/2 + m - j - 1 + \gamma)|z|^{2(j-1)}\Delta_k^j p + \sum_{j=0}^{[m/2]-1} c_{j,m}|z|^{2j}\Delta_k^{j+1}p \\
&= \sum_{j=1}^{[m/2]} c_{j,m}4j(N/2 + m - j - 1 + \gamma)|z|^{2j-2}\Delta_k^j p + \sum_{j=1}^{[m/2]} c_{j-1,m}|z|^{2j-2}\Delta_k^j p.
\end{aligned}$$

Using the expression of the constants $c_{j,m}$ given above , one can check that

$$4j(N/2 + m - j - 1 + \gamma)c_{j,m} + c_{j-1,m} = 0,$$

and therefore $\Delta_k(Q_0 p) = 0$, i.e. $Q_0 p \in \mathscr{H}_m$. Now, consider the following sequence of polynomials $\Delta_k^t p$ with $t = 0, \dots, [m/2]$. Since $p \in \mathscr{P}_m$, then $\Delta_k^t p \in \mathscr{P}_{m-2t}$ and, by the above discussion, $Q_0(\Delta_k^t p) \in \mathscr{H}_{m-2t}$ with

$$Q_0(\Delta_k^t p) = \sum_{j=0}^{[m/2]-t} c_{j,m-2t}|z|^{2j}\Delta_k^{t+j}p := Q_t p.$$

After multiplying both sides by $|z|^{2t}$, we obtain

$$\begin{aligned} |z|^{2t}Q_t p &= \sum_{j=0}^{[m/2]-t} c_{j,m-2t}|z|^{2(j+t)}\Delta_k^{j+t}p \\ &= \sum_{s=t}^{[m/2]} c_{s-t,m-2t}|z|^{2s}\Delta_k^s p. \end{aligned}$$

Write $a_{s,t} := c_{s-t,m-2t}$ with $t \le s \le [m/2]$ and $t = 0, \dots, [m/2]$. The above equation can be written as

$$\begin{bmatrix} a_{0,0} & a_{1,0} & a_{2,0} & a_{3,0} & \cdots \\ 0 & a_{1,1} & a_{2,1} & a_{3,1} & \cdots \\ 0 & 0 & a_{2,2} & a_{3,2} & \cdots \\ 0 & 0 & 0 & a_{4,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} p \\ |z|^2\Delta_k p \\ \vdots \\ |z|^{2[m/2]}\Delta_k^{[m/2]}p \end{bmatrix} = \begin{bmatrix} Q_0 p \\ |z|^2 Q_1 p \\ \vdots \\ |z|^{2[m/2]}Q_{[m/2]}p \end{bmatrix},$$

where the expressed upper-triangular matrix is invertible. By solving this system in p, we obtain

$$p = \sum_{t=0}^{[m/2]} \alpha_t |z|^{2t} Q_t p \tag{5.4}$$

where the constants α_t are independent of p. We claim that this decomposition is unique. Substituting in (??) the polynomial p by $|z|^{2\ell}Q_\ell p$ with $\ell = 0, \cdots, [m/2]$, we get

$$|z|^{2l}Q_\ell p = \sum_{t=0}^{[m/2]} \alpha_t |z|^{2t} Q_t(|z|^{2\ell}Q_\ell p). \tag{5.5}$$

By the uniqueness of the decomposition, as we claimed above, all the terms on the right hand side of (??) vanish only for $t = \ell$. Therefore

$$Q_\ell p = \alpha_\ell Q_\ell(|z|^{2\ell}Q_\ell p) \tag{5.6}$$

$$= \alpha_\ell \sum_{j=0}^{[m/2]-\ell} c_{j,m-2\ell}|z|^{2j}\Delta_k^{\ell+j}(|z|^{2\ell}Q_\ell p). \tag{5.7}$$

On the other hand, using (??) and the fact that $Q_\ell p \in \mathscr{H}_{m-2\ell}$, we obtain

$$\begin{aligned}
\Delta_k^\ell(|z|^{2\ell}Q_\ell p) &= \Delta_k^{\ell-1}\left[\Delta_k(|z|^{2\ell}Q_\ell p)\right] \\
&= \Delta_k^{\ell-1}\left[4\ell(N/2+\ell-1+m-2\ell+\gamma)|z|^{2(\ell-1)}Q_\ell p\right] \\
&= \Delta_k^{\ell-1}\left[4\ell(N/2-1+m-\ell+\gamma)|z|^{2(\ell-1)}Q_\ell p\right] \\
&= \Delta_k^{\ell-2}\left[4\ell(N/2+m-\ell-1+\gamma)4(\ell-1)(N/2+m-\ell-1+\gamma+1)]|z|^{2(\ell-2)}Q_\ell p\right] \\
&\vdots \\
&= 4^\ell\Gamma(\ell+1)\frac{\Gamma(N/2+m+\gamma-1)}{\Gamma(N/2+m-\ell+\gamma-1)}Q_\ell p.
\end{aligned}$$

Therefore, (??) becomes

$$\begin{aligned}
Q_\ell p &= \alpha_\ell 4^\ell\Gamma(\ell+1)\frac{\Gamma(N/2+m+\gamma-1)}{\Gamma(N/2+m-\ell+\gamma-1)}\sum_{j=0}^{[m/2]-\ell} c_{j,m-2\ell}|z|^{2j}\Delta_k^j(Q_\ell p) \\
&= \alpha_\ell 4^\ell\Gamma(\ell+1)\frac{\Gamma(N/2+m+\gamma-1)!}{\Gamma(N/2+m-\ell+\gamma-1)}c_{0,m-2\ell}Q_\ell p.
\end{aligned}$$

If $Q_\ell p \neq 0$, then

$$\alpha_\ell = \frac{\Gamma(N/2+m-\ell+\gamma-1)}{4^\ell\Gamma(\ell+1)\Gamma(N/2+m+\gamma-1)c_{0,m-2\ell}},$$

and the theorem holds. Next, we will prove our claim about the uniqueness of the decomposition. Assume that there exist two functions $\mathbb{P}^{(1)}_{m-2j}, \mathbb{P}^{(2)}_{m-2j} \in \mathscr{H}_{m-2j}$ such that

$$p = \sum_{j=0}^{[m/2]} |z|^{2j}\mathbb{P}^{(1)}_{m-2j} = \sum_{j=0}^{[m/2]} |z|^{2j}\mathbb{P}^{(2)}_{m-2j}.$$

Therefore

$$\sum_{j=0}^{[m/2]} |z|^{2j}\mathbb{P}^{(0)}_{m-2j} = 0, \qquad \mathbb{P}^{(0)}_{m-2j} = \mathbb{P}^{(1)}_{m-2j} - \mathbb{P}^{(2)}_{m-2j}. \tag{5.8}$$

After applying $\Delta_k^{[m/2]}$ to (??) and using the fact that $\mathbb{P}^{(0)}_{m-2j} \in \mathscr{H}_{m-2j}$, we obtain

$$\begin{aligned}
0 &= \sum_{j=0}^{[m/2]} \Delta_k^{[m/2]}(|z|^{2j}\mathbb{P}^{(0)}_{m-2j}) \\
&= \sum_{j=0}^{[m/2]} \Delta_k^{[m/2]-1}\Big[4j(N/2+j-1+m-2j+\gamma)|z|^{2(j-1)}\mathbb{P}^{(0)}_{m-2j} + |z|^{2j}\Delta_k\mathbb{P}^{(0)}_{m-2j}\Big] \\
&= \sum_{j=1}^{[m/2]} 4j(N/2+m-j-1+\gamma)\Delta_k^{[m/2]-1}(|z|^{2(j-1)}\mathbb{P}^{(0)}_{m-2j})) \\
&= \sum_{j=2}^{[m/2]} 4^2j(j-1)(N/2+m-j+\gamma-1)(N/2+m-j+\gamma-2)\Delta_k^{[m/2]-2}(|z|^{2(j-1)}\mathbb{P}^{(0)}_{m-2j}) \\
&= 4^{[m/2]}\Gamma(1+[m/2])\frac{\Gamma(N/2+m+\gamma-[m/2])}{\Gamma(N/2+m-2[m/2]+\gamma)}\mathbb{P}^{(0)}_{m-2j}.
\end{aligned}$$

Therefore $\mathbb{P}^{(0)}_{m-2j} = 0$ and (??) becomes

$$\sum_{j=0}^{[m/2]-1} |z|^{2j}\mathbb{P}^{(0)}_{m-2j} = 0.$$

Now, we Apply $\Delta_k^{[m/2]-1}$ to the above equation and we obtain $\mathbb{P}^{(0)}_{m-2([m/2]-1)} = 0$. The same argument gives $\mathbb{P}^{(0)}_{m-2j} = 0$ for all $0 \le j \le [m/2]$, and the uniqueness of the decomposition holds. □

For $g \in G$, denote by $\pi(g)$ the left regular action of G on $\mathscr{F}_k(\mathbb{C}^N)$

$$\pi(g)f(z) = f(g^{-1}z).$$

The infinitesimal representation $d\pi$ commutes with ω.

For fixed $h \in \mathscr{H}_m$, let $\mathscr{I}h := \{|z|^{2t}h,\ t = 0, 1, \ldots\}$. Since $g \circ \Delta_k \circ g^{-1} = \Delta_k$, the space $\mathscr{I}h$ is invariant under the action of G. Also, the representation ω leaves $\mathscr{I}h$ invariant. Indeed, $H(|z|^{2t}h) = (m + 2t + N/2 + \gamma)|z|^{2t}h$, $E(|z|^{2t}h) = 1/2|z|^{2(t+1)}h$, and by (??)

$$\begin{aligned}
F(|z|^{2t}h) &= -1/2\Big[4t(m+t+N/2+\gamma-1)|z|^{2(t-1)}h + |z|^{2t}\Delta_k h\Big] \\
&= -2t(m+t+N/2+\gamma-1)|z|^{2(t-1)}h.
\end{aligned}$$

We summarize the consequences of all the above computations and dissections in the light of Theorem ??.

Theorem 5.2. *As a* $G \times \mathfrak{sl}(2,\mathbb{R})$*-module, the Fock space admits the following multiplicity-free decomposition*

$$\mathscr{F}_k(\mathbb{C}^N) = \bigoplus_{m=0}^{\infty} \mathscr{H}_m \otimes \mathscr{W}_{m+N/2+\gamma},$$

where $\mathscr{W}_{m+N/2+\gamma}$ is the representation with lowest weight $m+N/2+\gamma$. We also have the separation of variables theorem providing the following $G \times \mathfrak{sl}(2,\mathbb{R})$ decomposition

$$\mathscr{P}(\mathbb{C}^N) = \sum_{m=0}^{\infty}{}^{\oplus} \sum_{t=0}^{[m/2]}{}^{\oplus} |z|^{2t}\mathscr{H}_{m-2t}.$$

The following is then immediate.

Corollary 5.3. *Under the action of $\mathfrak{sl}(2,\mathbb{R})$, the Fock space $\mathscr{F}_k(\mathbb{C}^N)$ decomposes as*

$$\mathscr{F}_k(\mathbb{C}^N) = \bigoplus_{m=0}^{\infty} \dim(\mathscr{H}_m)\mathscr{W}_{m+N/2+\gamma},$$

where $\dim(\mathscr{H}_m) = \begin{pmatrix} m+N-1 \\ N-1 \end{pmatrix} - \begin{pmatrix} m+N-3 \\ N-1 \end{pmatrix}$. If $N>1$, this is always nonzero, but if $N=1$, it is zero for $m \geq 2$.

As an application of the above demonstrated $\mathfrak{sl}(2,\mathbb{R})$-representation theory, we obtain the Hecke's formula for the Dunkl transform as following: Recall that $H = \mathscr{B}_k \circ \left[\frac{1}{4}(-\Delta_k + 4|x|^2)\right] \circ \mathscr{B}_k^{-1}$ (see Theorem ??), where $\mathscr{B}_k^{-1}$ can be written as (see Corollary ??)

$$\mathscr{B}_k^{-1} = 2^{\gamma+N/2} c_k^{-1/2} e^{-|x|^2} d_2 \circ e^{-\Delta_k/2}.$$

Therefore

$$\left[\frac{1}{4}(-\Delta_k + 4|x|^2)\right] \mathscr{B}_k^{-1}(p) = (m + N/2 + \gamma)\mathscr{B}_k^{-1}(p), \qquad p \in \mathscr{P}_m(\mathbb{C}^N).$$

Notice that, for all $p \in \mathscr{H}_m$, we have $\mathscr{B}_k^{-1}(p) = 2^{\gamma+N/2} c_k^{-1/2} e^{-|x|^2} p$, which implies that $e^{-|x|^2}p$ is an eigenvector for $\left[\frac{1}{4}(-\Delta_k + 4|x|^2)\right]$ with eigenvalue $(m + N/2 + \gamma)$. On the other hand, $\left[\frac{1}{4}(-\Delta_k + 4|x|^2)\right]$ is the generator of the Lie algebra $\mathfrak{k} \cong \mathfrak{so}(2)$, while the Dunkl transform $\mathscr{D}_k$ can be written as (see Corollary ??)

$$\mathscr{D}_k = e^{i\frac{\pi}{2}(\gamma+N/2)} e^{-\frac{\pi}{8}(-\Delta_k + 4|x|^2)}$$

Hence, for $p \in \mathscr{H}_m$

$$\begin{aligned} \mathscr{D}_k(e^{-|x|^2}p) &= e^{i\frac{\pi}{2}(\gamma+N/2)} e^{-\frac{\pi}{2}\mathfrak{k}}(e^{-|x|^2}p) \\ &= e^{i\frac{\pi}{2}(\gamma+N/2)} e^{-i\frac{\pi}{2}(m+N/2+\gamma)} e^{-|x|^2} p \\ &= e^{-i\frac{\pi}{2}m} e^{-|x|^2} p, \end{aligned}$$

and the following theorem stands.

Theorem 5.4. *The following Hecke's type formula holds*

$$\mathscr{D}_k(e^{-|x|^2}p) = e^{-i\frac{\pi}{2}m} e^{-|x|^2} p, \qquad p \in \mathscr{H}_m.$$

6. The Weyl quantization map and the Berezin transform

In this short section, we will use the restriction principle to set the Berezin transform, and, abstractly, the Weyl quantization map, for a Coxeter group G.

Consider $\mathscr{F}_k(\mathbb{C}^N)\otimes\overline{\mathscr{F}_k(\mathbb{C}^N)}$ realized as the space of Hilbert-Schmidt operators on $\mathscr{F}_k(\mathbb{C}^N)$. Each Hilbert-Schmidt operator $\mathbb{T}$ on $\mathscr{F}_k(\mathbb{C}^N)$ is given by a kernel $F(z,w)$ holomorphic in z and anti-holomorphic in w with $\mathbb{T}f(z) = \langle\!\langle f, \overline{F(z,\cdot)}\rangle\!\rangle_k$ and $\|\mathbb{T}\|_{HS} = \|F\|_{\mathscr{F}_k\otimes\mathscr{F}_k}$. Henceforth, we will identify the operator with its kernel.

To realize the Weyl quantization map, we define

$$\begin{aligned}
&\mathfrak{R}_k : \mathscr{F}_k(\mathbb{C}^N)\otimes\overline{\mathscr{F}_k(\mathbb{C}^N)} \to \mathscr{C}^\infty(\mathbb{R}^{2N})\\
&\mathfrak{R}_k F(x,y) = F(x+iy, x+iy)e^{-(|x|^2+|y|^2)}, \qquad x,y\in\mathbb{R}^N.
\end{aligned}$$

Clearly $\mathfrak{R}_k$ is injective and closed. Further, since the Gaussian $e^{-(|x|^2+|y|^2)}$ belongs to $\mathscr{L}^1(\mathbb{R}^N, w_k)\otimes \mathscr{L}^1(\mathbb{R}^N, w_k)$ and $\mathscr{F}_k(\mathbb{C}^N)\otimes\overline{\mathscr{F}_k(\mathbb{C}^N)}$ contains all polynomials, the range of $\mathfrak{R}_k$ contains the functions of the form $e^{-(|x|^2+|y|^2)}p(x+iy, x+iy)$ where $p(z,z)$ are polynomials of x and y.

Now, consider the formal adjoint $\mathfrak{R}_k^* : \mathscr{L}^2(\mathbb{R}^N, w_k)\otimes\mathscr{L}^2(\mathbb{R}^N, w_k) \to \mathscr{F}_k(\mathbb{C}^N)\otimes\overline{\mathscr{F}_k(\mathbb{C}^N)}$. Employing the same argument used in the proof of Proposition 4.1, and the fact that $\mathbb{K}_z\otimes\overline{\mathbb{K}_w}$ is the reproducing kernel of $\mathscr{F}_k(\mathbb{C}^N)\otimes\overline{\mathscr{F}_k(\mathbb{C}^N)}$, we can write $\mathfrak{R}_k^*$ as

$$\mathfrak{R}_k^* f(z,w) = \iint_{\mathbb{R}^N\times\mathbb{R}^N} f(x,y)\mathbb{K}(x+iy, z)\overline{\mathbb{K}(x+iy,w)}e^{-(|x|^2+|y|^2)}w_k(x)w_k(y)\,dx\,dy.$$

One can think of $\mathfrak{R}_k^*$ as the Wick quantization map.

Form the adjoint operator $\mathfrak{R}_k\mathfrak{R}_k^*$ on $\mathscr{L}^2(\mathbb{R}^N, w_k)\otimes\mathscr{L}^2(\mathbb{R}^N, w_k)$. It is integral representation is given by

$$\mathfrak{R}_k\mathfrak{R}_k^* f(a,b) = \iint_{\mathbb{R}^N\times\mathbb{R}^N} f(x,y)|\mathbb{K}(x+iy, a+ib)|^2 e^{-(|x|^2+|a|^2)}e^{-(|y|^2+|b|^2)}w_k(x)w_k(y)\,dx\,dy.$$

The transformation $\mathfrak{R}_k\mathfrak{R}_k^*$ is the Berezin transform related to G.

Theorem 6.1. (cf. [9, Theorem XII.7.6.7]) *If T is a closed transformation whose domain is dense, then T can be written in one and only one way as a product $T = PA$, where P is a partial isometry and A is a positive self adjoint transformation.*

Let

$$\mathfrak{R}_k^* = \mathfrak{B}_k\sqrt{\mathfrak{R}_k\mathfrak{R}_k^*}$$

be the polar decomposition of $\mathfrak{R}_k^*$. The map $\mathfrak{B}_k$ is the Weyl quantization map. The properties of $\mathfrak{R}_k$ together with Theorem 5.1 imply the following:

Theorem 6.2. *The Weyl quantization map $\mathfrak{B}_k$ is a unitary operator from $\mathscr{L}^2(\mathbb{R}^N, w_k)\otimes\mathscr{L}^2(\mathbb{R}^N, w_k)$ to $\mathscr{F}_k(\mathbb{C}^N)\otimes\overline{\mathscr{F}_k(\mathbb{C}^N)}$.*

Remark 6.3. One may define the restriction map $\mathfrak{R}_k$ by $\mathfrak{R}_k F(x,y) = F(x+iy, x+iy)e^{-\omega(|x|^2+|y|^2)}$ with $\omega > 1$. Therefore, one can show that $\|\mathfrak{R}_k\| \leq c_k(4(\omega-1))^{-(\gamma+N/2)}$, i.e. $\mathfrak{R}_k$ is a bounded operator. However, from this definition of $\mathfrak{R}_k$ it is too rough to obtain the transformation $\mathfrak{B}_k$ as the deformation of the classical Weyl transform.

Observe that the Dunkl-kernels in dimension 2N and N are related by

$$E_k((a,b),(x,y)) = E_k(a,x)E_k(b,y), \qquad (a,b),(x,y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

For $f \in \mathscr{L}^2(\mathbb{R}^N, w_k) \otimes \mathscr{L}^2(\mathbb{R}^N, w_k)$, put

$$\mathscr{D}_k^{\otimes} f(\xi,\eta) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} f(x,y)E_k(-i\xi,x)E_k(-i\eta,y)w_k(x)w_k(y)dxdy.$$

Corollary 6.4. *The map* $F \mapsto \mathscr{D}_k^{\otimes}(\mathfrak{B}_k^* F)$ *is a unitary operator from* $\mathscr{F}_k(\mathbb{C}^N) \otimes \overline{\mathscr{F}_k(\mathbb{C}^N)}$ *onto* $\mathscr{L}^2(\mathbb{R}^N, w_k) \otimes \mathscr{L}^2(\mathbb{R}^N, w_k)$.

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